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ABSTRACT

A two-phase study was designed to investigate the relationship between cognitive skills hypothesized (by Piaget) to underlie number competence, and performance on tasks requiring logical reasoning with number-related concepts. During the first phase, a battery of tasks was administered to 60 kindergarten and 60 third-grade students. These tasks were designed to assess acquisition of concepts identified as formally related to various aspects of number concepts. Tasks within a concept area were weighted according to their relative importance and degree of difficulty. Data were analyzed using contingency tables. In the second phase of the study, data were reanalyzed using a set of behavioral indices developed by Brainerd in earlier work. The results of the study support Piagetian theories of cognitive development. (SD)

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Technical Report No. 340

A STUDY IN THE NATURE AND DEVELOPMENT OF THE NATURAL NUMBER CONCEPT: INITIAL AND SUPPLEMENTARY ANALYSES

by Arthur J. Gonchar

Report from the Project on
Children's Learning and Development

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ABSTRACT

The present two-phase study sought to investigate the structural isomorphic relations between the cognitive skills which presumably underlie number competence as formalized by Piaget (1966) and the actual performance patterns of children when confronted with task problems requiring logical reasoning and involving number-related concepts for their solutions. In the main study, the accuracy of Piaget's predictions concerning the developmental priority and interpatterning of relations between emergent cognitive competencies of classes and relations was assessed. Since his theory contends that number is in co-development with and a resultant of the coordinated synthesis of classes and relations, measures of the developmental relatedness and concurrence of conceptual performance patterns were applied in assessing the appropriateness of Piaget's model. In the second phase of this investigation, an attempt was made to reconcile Piaget's model and the recent experimental evidence presented by Brainerd (1972, 1973a, 1973b, 1973c, 1974) in support of an ordinal sequence in the development of the natural number concept.

Initial discussion focused on the study's attempt to corroborate the philosophical and observational techniques of inquiry in psychological investigations by (a) presenting by detailed theoretical explication the differing interpretations of natural number as viewed in the philosophies of intuitionism, logicism, and formalism and (b) describing the psychological role of traditional logic in identifying and characterizing the elements in developmental processes.

In the main study, two assessment situations for each concept were adapted (including the measures employed by Piaget, 1952; Kofsky, 1963; and Brainerd, 1973a) as those being the behavioral isomorphs of the formal definitions of natural number derived from mathematical philosophy and its partitioned elements of classes and relations. Tasks within a concept area were weighted as to their relative importance to the acquisition of the particular concept and to the differential degree of difficulty reflected in certain specified task features. In the second analysis, a set of behavioral indices similar to those chosen by Brainerd as appropriate in the study of natural number concept was employed. In both studies, behavioral indices of competence were specified and included within a total scoring scheme employing a standard criterion across concepts and a three-stage qualitative model of conceptual acquisition analogous to that developed by Piaget. The entire task battery was administered to a core group of 60 kindergarteners and 60 third graders as part of a four-year, longitudinal investigation (Hooper & Klausmeier, 1973) with annual assessments. Data from the initial year's assessment are reported here.

The results of the main study suggested a developmental picture in which the concepts of classes and relations evolve co-jointly with the natural number concept among which a universal and invariant sequence is not apparent. While 53 percent of the subject sample demonstrated synchronous development (concordant rankings on all three concepts) in reference to the number-related concepts, the remaining subjects demonstrated a number of alternative and potentially ambiguous sequential

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patterns. The relevant data consisting of triads of cognitive developmental acquisitions did not reveal a high degree of sequential regularity across same age samples or an invariant pattern between age levels. The natural number concept seems to develop through an extended process of concordant and discordant patterning. While some children acquired the set of cognitive elements in one consistent (co-emerging) pattern, others acquired it in a number of other patterns. This does not necessarily suggest the absence of any functional developmental connection among the elements of relations, classes, and number. The strong values of association registered between concepts, the high proportion of subjects with concordant rankings, and the lack of a clear developmental sequence suggested that there is a possible underlying affiliation between the concept of relations and the concept of classes in the construction of the natural number concept. In contrast, experiments using the tasks suggested by Brainerd result in confirming evidence for the ordinal theory of natural number development. It was argued that these effects reflect the differential difficulty of Brainerd's concept tasks and do not accurately assess developmental acquisitions.

In sum, the findings of this investigation are in agreement with Piaget's contention that cognitive development is dialectical in nature. Without further longitudinal follow-up, it would be conjecture to conclude definitively from the data that Piaget's model of number development (being the resultant of the synchronous emergence of relational and classification competencies) is upheld.

I

INTRODUCTION

The logic underlying the child's understanding of fundamental concepts has received considerable attention by epistemologist Jean Piaget. He maintains that the psychological activity of the human mind is essentially logical (the elements are lawfully bound in intercoordinated networks and regulated in their operation and transformation). It is this property of cognitive functioning that is key to understanding man's intellect and his understanding of nature. In his efforts to discover both the logical nature and ontogenetic patterns underlying the structure of cognitive development, Piaget first defines the scope and content of conceptual behavior in terms of primary components, with reference to the basic functional relations between them. He then studies the successive forms of this behavior at various age levels.

Logic, Piaget believes, is inherent in psychic activity from birth and evolves in the course of individual development. David Elkind captures the essence of Piaget's position in the following preface to the English edition of one of Piaget's works:

> The logic to be discerned in the behavior of the infant is much more primitive and much less systematic than that observed in the preschool or elementary school child. It is only in adolescents that a true, or formal, logical system, comparable to that constructed by logicians, develops. But if logic is taken in the broad sense to mean a set of actions that obey logic-like rules, such as transitivity, then it can be said that at all age levels behavior manifests some form of logic [Elkind, 1967, p. x].

Piaget further asserts that logic is inherent in human behavior and that its form changes with developmental sophistication. In many of his articles, Charles Brainerd introduces his psychological investigations into the interpatterning of mental capacities with discussions of and concern for the logic of formal mathematical philosophy. "It seems reasonable to suppose that the elements of cognition ('images,' 'engrams,' etc.) and the relations which serve to unite them ('thought,' 'operations,' etc.) are in the same sense isomorphic with the elements and relations of logic [Brainerd, 1973a, p. 79]." Brainerd believes, as does Piaget, that a theory adequate to explain the developmental processes in the acquisition of knowledge must of necessity incorporate an analysis dealing with the relationship between mathematical derivations of logic and the emergence of logical concepts in children's thinking. Such an analysis begins with a description, by logical descriptors, of the edifice of human knowledge--the conceptual elements, structure, and systems of transformations--and

culminates with the construction of a formal model based on naturalistic and experimental observations.

The traditional axioms offered by formal mathematical philosophy are held by Piaget to be too rigid and too atomistic to reflect the dialectical character of the operational systems involved in the development of logical reasoning competencies in the middle childhood period. In response to this apparent void in philosophical exposition, he and logician Jean-Blaise Grise (see Piaget, 1966) applied the elements and principles (primitive terms, axioms, and rules of inference) of mathematical logic to a psychogenetic analysis of cognitive events. This process resulted in the construction of "psycho-logic" which in their view parallels the natural ontogenesis of human cognitive abilities.

While Piaget concedes that epistemological psychologists describe the hypothesized mental structures and operations by referring to models borrowed from elementary symbolic logic, he firmly asserts that this is a heuristic procedure which is used only to describe how knowledge is constructed. He does not seek to reduce the study of thought to a formal logical model:

> But it is important to make it clear (and to emphasize this strongly) that for us it was not a question of reducing natural thought to formal models, but the entirely different one, of using the most precise language possible to describe natural structures, making on the contrary a conscious effort to take account of the limitations proper to the latter and to arrive at the most rudimentary and most elementary possible kinds of structured wholes (without worrying about their lack of generality nor especially about their logical consistency) [Piaget, 1966, p. 169].

Various logical models have organized the notion of number into objective systems using abstract symbols and axioms. Philosophical treatments of number have differed in the primitive terms or postulates which form their basis. Their statements, presented to establish the existence of a set of physical and psychological elements in abstract form as well as the property of the relations between elements, the operations of the elements, and the properties of those operations, have led to diverse theoretical routes of deductions and have yielded disparate conclusions about the origin and derivation of number. Piaget has accumulated facts about the acquisition and successive forms of the natural number concept in human cognitive development through a process of observation and experimentation. His model applied to natural number concept development is offered as a parsimonious package arriving at the same conclusions about the nature and acquisition of natural number knowledge by two different but complementary routes: one by logical argument and the other by observation and experimentation.

This study investigates the structural isomorphic relations between the cognitive skills which presumably underlie number competence as formalized by Piaget (1966) and the actual performance patterns of children when confronted with task problems that involve number-related concepts for their solutions. Essentially, the question posed in this study is whether in the spontaneous coordination of the child's action, when he manipulates

stimulus impressions, he employs structural elements analogous to those hypothesized by Piaget's philosophy. The accuracy of Piaget's predictions concerning the developmental priority and interpatterning of relations between emergent cognitive competencies of classes and relations is central to the analysis and discussion of relevant empirical data. Since his theory contends that number is in co-development with and a resultant of the coördinated synthesis of classes and relations, measures of the developmental relatedness and concurrence of conceptual performance patterns will be applied in evaluating the appropriateness of Piaget's model.

Initial discussions focus on the study's attempt to corroborate the philosophical and observational techniques of inquiry in psychological investigations by (a) presenting by detailed theoretical explication the differing interpretations of natural number as viewed in the mathematical philosophies of intuitionism, logicism, and formalism and (b) describing the psychological role of traditional logic in identifying and characterizing the elements in the developmental processes. Assessment situations for each concept were adapted as behavioral isomorphs to the formal definitions of natural number derived from mathematical philosophy and its partitioned elements of classes and relations. Once specified the behavioral indices of competence were included within a total scoring scheme employing a standard criterion across concepts and a three-stage qualitative (global-perceptual, intuitive-intensional, and operational-extensional) model of conceptual acquisition analogous to that developed by Piaget. Tasks within a concept area are weighted as to their relative importance to the acquisition of the particular concept and to the differential degree of difficulty reflected in certain specified task features.

The task battery was administered to subjects drawn from the middle childhood period (60 kindergarteners and 60 third graders). Hypotheses relating to acquisition patterns for related number concepts were investigated using the McNemar's Test for Equality of Correlated Proportions. The Cochran Q technique was used to assess the degree of concordance across concept area performance. The Goodman-Kruskal Gamma statistic (as suggested by Wohlwill, 1973, pp. 218-236) was used to assess the association and consistency of concordant concept performance across same-age subjects.

INSTRUCTIONAL IMPLICATIONS

There are many implications of epistemological research seen from the Piagetian perspective. It is by studying the natural growth of thought in psychological terms that a structure may be discovered and delineated. Natural structure, governed by internal principles of logic, may be determined to give coherence and form to thought, thus providing a basis for rationality. Discovering the laws and mechanisms which govern and regulate the operation and maturity of conceptual processes improves man's potential to understand and influence the course and content of human intellectual development. Consider now the emphasis placed on mathematical understanding of natural number as reported by Charles Brainerd.

The concept of number has been of social importance since the beginning of recorded history. The significance to society of number and

number-related skills has increased tremendously with the rise of industrial civilization. In most Western nations today children receive considerable exposure to number concepts early in life. More or less haphazard at first, the exposure becomes simultaneously more intense and more systematic with the onset of formal education. During the first few years of elementary school roughly 50 percent of the curriculum is normally given over to inculcating the natural numbers and methods of manipulating them. We expect that by adolescence children will possess a numerical competence much higher than that of an educated Greek or Roman adult of two millenniums ago. We have even developed labels that imply mental turpitude on the part of those children who fail to attain the standard of numerical competence that we deem desirable, for example, "learning disability" and "underachievement" (Brainerd, 1973b, p. 104).

Because modern society has placed so much emphasis on number competence, it is imperative to have extensive information on how the concept of number unfolds naturally in the course of a child's mental growth. This information can be applied to mathematics curricula by determining the units, structural relationships, and sequencing which best serve to facilitate the development and learning of number concepts and skills. The curriculum specialist who understands the processes involved in the construction of mathematical knowledge can better adapt a child's mathematical experiences in school to his developing logical sense of how the world operates.

With the data obtained from this study educators could design, order, and pace a curriculum of sequential learning hierarchies that would more nearly match the child's natural construction of number concepts. With the units of number concepts clearly defined and proven essential to the construction of mathematical knowledge, experiences which directly relate to these units may be provided to the child.

A word of caution is essential at this juncture. The statements about instructional implications above assume that a curriculum which closely parallels the natural sequence found in young children will facilitate the child's development. I submit that the question educational psychologists ask of themselves ought not be, "How can we facilitate or accelerate development?" but rather, "How may we develop instructional programs and provide an environment which would do the least amount of harm to children?" Viewed in this way the essential function of the educator becomes one of providing and organizing educational experiences which more closely conform to and are supportive of children's natural developmental patterns.

Since educators continue to provide mathematical experiences for the school child, it is imperative that they be provided with information concerning the way the child naturally constructs mathematical concepts and how logical-mathematical thought parallels well-defined principles and laws (Scandura & McGee, 1972; Wang, Resnick, & Boozer, 1971).

II

PHILOSOPHY OF NUMBER

DEFINITION

Before proceeding further with either a philosophical or psychological treatment of natural number, it is imperative to understand what is meant by the term. In a broad sense, natural number may be conceived of as the series of positive integers which is the early part of an infinite series and corresponds to a certain group of everyday concepts expressed in language by the number symbols "one," "two," "three," etc., either in speech or writing. Unlike the other number systems (e.g., irrational numbers or complex numbers) which are theoretical abstractions derived from it, the natural number system is concrete, and represents real entities in the physical world. The numbers 1, 2, 3, 4, 5 . . . are called natural numbers because it is generally felt that they have in some philosophical sense a natural existence independent of man. The more complicated of the number systems, in contrast, are regarded as intellectual constructs of man's invention.

OVERVIEW

To date there is no universally accepted philosophical theory to explain the mathematical basis of natural numbers. Even the fundamental question of whether numbers can be reduced to their component elements remains unresolved. The philosophers Poincare, Browver, and the school of intuitionists maintain that number cannot be reduced to separable logical entities. They returned to the Pythagorean position that the natural numbers must be accepted as given without further analysis. This position is in opposition to two other schools of modern mathematical thought, logicism and formalism, which maintain that numbers can be logically distilled and considered on the basis of their most primitive properties. These schools maintain that mathematics is a part of logic and that natural numbers, being a part of mathematics, are definable in terms of a very small number of fundamental logical concepts. Although the latter two schools agree that the concept of number can be divided into more fundamental notions, they disagree about the relative primacy of these notions. This examination will focus on these latter two schools of mathematical thought, and a third philosophical-psychological interpretation presented by Piaget.

The three models that will be analyzed in this investigation are (1) Peano's five axioms, a theory which makes use of the concepts of transitive asymmetrical relations and numbers as a series of successors and is favored by the formalists; (2) Russell's formulations which arrive at the conclusion that number is essentially classes of classes--a theory which is supported by the logicists, and (3) the empirical conclusions presented by Piaget and mathematically formulated by Grize which hold number to be the product of both classes and relations simultaneously.

These three accounts of the primitive foundations of the concept of number differ in their essential characteristics. The ordinal position of the formalists is based on viewing the entire natural number system as a series. The cardinal position supported by the logicists focuses on the individual numbers separately and is based on the property of class membership. "Piaget's account emphasized both the holistic and atomistic properties of the other two theories . . . . Piaget's particular theory of number becomes a direct reflection of his general theory of the nature of human knowledge . . . the part and the whole are epistemologically inseparable [Brainerd, 1973a, p. 33]."

THE ORDINAL POSITION

Demonstrating that all of mathematics is forged on the base of the natural number system, Peano constructed his theory of number on the mathematically undefined concepts of success (order), zero (its origin), and number itself (a vacuous place holder). Note, then, the theme of ordinality and number as an ordered progression in Peano's five postulates:

Postulate I:	0 is a number.
Postulate II:	The successor of any number is a number.
Postulate III:	No two numbers have the same successor.
Postulate IV:	0 is not the successor of any number.
Postulate V:	Any property which belongs both to 0 and also to the successor of every number which has the property, belongs to every number.

It follows from Peano's propositions that natural number is basically ordinal in nature: inherent is its property of order. Innate meaning can only be understood by the particular manner in which it is arranged--a continuous series of successors, giving rise to an endless series of new numbers. Peano also recognized that number does not derive its entire meaning from order; an element of quantification is also required for understanding it. Natural number is considered a set of finite integers, a string of symbols that are ascribed values representing concrete quantitative entities in the physical world while retaining its primarily ordinal basis. Accordingly, number is then defined as a quantifiable ordered progression, a series of ascending or descending asymmetric quantitative relationships. For example, given three elements x, y, z and the existence of an asymmetric quantitative relationship between them "R," which serves to distinguish them apart, if x R y and y R z hold, then x R z also holds. That is, if R holds between the first and second terms and between the second and third terms, then it also holds between the first and third terms. This is the "minimum ordinal proposition" and the basis of the ordinal system of number.

THE CARDINAL POSITION

Logicism, a philosophical system first formulated by Leibnitz and taken up and furthered by Frege, Russell, and Whitehead, maintains a

different position on the question of the element of natural number which serves as its most primitive property. Russell defines number by its property of being a set of similar classes. He is critical of Peano's theory because the notion of ordered progression can be generalized to any environmental event and is not specific to natural number. He claims Peano's definition overlooks the most important property of natural number, classes, and says that Peano placed undue emphasis on the ordinal nature of number. In short, "Each natural number should be viewed as a class of equivalent classes . . . . Order simply is not all that primitive a notion and, therefore, it is not necessary to base the theory of natural numbers on order [Brainerd, 1973a, p. 22]." "In counting, it is necessary to take the objects counted in a certain order, as first, second, third, etc., but order is not the essence of number, it is an irrelevant addition, an unnecessary complication from the logical point of view [Russell, 1917, p. 17]."

Russell sees number as a gathering together into a single aggregate all the collections that contain the same quantity of elements. Number groups collections, i.e., all collections of two elements into one group, all collections of three members into another, and so on. In this manner, what are obtained are various groupings of collections, each group consisting of all the collections that have a certain quantity of elements (Russell, 1917, p. 14).

Instead of referring to a collection of members, the label "class" is substituted. A number can be considered a class whose members are similar collections, or a class of classes. To further explain Russell's concept of number, the following example is presented. Three is the number representing the group of all collections of three members; thus, three is the class of all classes possessing the cardinal property of threeness. The cardinal value ascribed to a given class is the set of all those classes that are similar to the given classes of members. The number three is something which all trios have in common and which distinguishes them from other collections (Russell, 1917, p. 12). Number allows any collection of members to be placed into a class in which are other collections that are similar in numerosity to it. Thus for Russell, a number is something that characterizes certain collections, namely, those that have that number. In sum, then, from the logicist's perspective, number is cardinal (classes of similar classes) in nature and is not predicated on the notion of ordinality or of series.

THE ORDINAL-CARDINAL POSITION

An alternative model is presented in the collaborative formalization of the logician Jean-Blaise Grize and the epistemologist Jean Piaget (see Piaget in Beth & Piaget, 1966). They argue that the natural process observed in the child's construction of number may correspond to a formal logical construction which explains the concept of natural number as a synthesis of overlapping notions of classes and relations. Piaget finds that neither logicism nor formalism accounts for the actual development of natural number concepts in humans. Based on clinical observations he concludes that the concept of natural number is formed precisely at

the intellect level at which the logic of relations and classes appears. He asserts that number is at the same time both a class and an ordered progress and that it is derived not from one or the other of these logical entities but from their union. It is "a simultaneous construction of the structures of classes, relations, and number [Piaget, 1966, p. 259]." Both the ordinal premise of Peano and the cardinal-based number concepts of Russell are unacceptable, not because of any inherent deficiency in their reasoning, but solely in terms of their psychological utility. They fail to state in their perspectives the mutual dependency and interrelatedness of classes and relations.

Piaget argues that both the ordinal and cardinal formalizations of natural number make use of classes and relations together but often in a form which is more implicit than explicit. "If we look at any theoretical formulation of number, we will find that in the number theories based on ordination there is always an element of inclusion (classes). Similarly, in theories based on cardination there is always an element of order [Piaget, 1970a, p. 39]." Piaget contends that these perspectives refer to only one of these two logical entities and reintroduce the other "almost surreptitiously, later, in the guise of an expository device or construction [Piaget, 1966, p. 272]." Peano deals with natural number as a series and introduces quantification later. Russell refers only to classes in his construction and introduces order later as an "unessential" component.

Piaget warns his reader not to consider his theory of natural number, with its emphasis on the interdependence of classes, relations, and natural number, as an intuitionist's viewpoint which holds number to be irreducible to its component entities (as in the classic Pythagorean viewpoint). To substantiate his belief, he cites evidence of parallelism between the evolution of number and developing notions of classes and relations. Piaget claims that the parallel construction of these "structures" is the "first piece of evidence in favour of their interdependence as against the view that there is an initial autonomy of number [Piaget, 1966, p. 261]."

In the development of human conceptualization, Piaget asserts, number is organized stage after stage, in close connection with the gradual elaboration of systems of inclusions (a hierarchy of logical classes) and systems of asymmetrical relations (qualitative seriation). The sequence of natural number construction is the result of an operational synthesis and coordination of classification and seriation.

> When we study the development of the notion of number in children's thinking, we find that it is not based on classifying operations alone but that it is a synthesis of two different structures. We find that along with the classifying structures, which are an instance of the Bourbaki algebraic structures, number is also based on ordering structures, that is, a synthesis of these two different types of structures. It is certainly true that classification is involved in the sense that two is included in three, three is included in four, etc. But we also need order relationships, for this reason: if we consider the elements of the classes to be equivalent (and this of course is the basis of the notion of number), then by this very fact

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it is impossible to distinguish one element from another--it is impossible to tell the elements apart. We get the tautology A + A = A; we have a logical tautology instead of a numerical series. Given all these elements, then, whose distinctive qualities we are ignoring, how are we going to distinguish among them? The only possible way is to introduce some order. The elements are arranged one after another in space, for instance, or they are considered one after another in time, or they are counted one after another. This relationship of order is the only way in which elements, which are otherwise being considered as identical, can be distinguished from one another.

In conclusion, then, number is a synthesis of class inclusion and relationships of order. It depends on an algebraic type of structure and an ordering type of structure, both at one time. One type of structure alone is not adequate.

I think that it is really quite obvious, if not trite, that number is based on two different types of operation. In fact, if we look at any theoretical formulation of number, we will find that in the number theories based on ordination there is always an element of inclusion. Similarly, in theories based on cardination there is always an element of order [Piaget, 1970a, pp. 38-39].

Piaget constructs his notion of the natural number system on Bourbaki algebraic structures, mathematical networks, and topology. He gives equal prominence to both cardinal and ordinal notions. In Piaget's view number is a system of hierarchical classifications, a series of inclusion relationships in which each successive number constitutes a class which includes all its predecessors. For example, the number 5 includes 4, 3, 2, 1; the number 4 includes 3, 2, 1; and so on. The number 5 is not just a simple composition of classes of 4, 3, 2, 1, for simultaneously the number 5 is a series of asymmetrical relations 5 > 4 > 3 > 2 > 1. Number is, therefore, a series of classes in which any one class is greater than any of those that precede it and less than any of its successors. Number could not be understood without some reference to order, implicit if not explicit. If the qualities distinguishing the individual elements within a collection were disregarded, there would be no way to obtain that collection's numerosity (its count, therefore, its numerical classification). The only possible way to make a distinction between the identical elements is to impose order on them.

The crux of Piaget's theory of natural number lies in the notion of "units," derived from the synthesis of inclusions. The elements of a collection must be considered units, existing as independent, individual entities at the same time that they exist as a collective aggregate bonded by a common property defining their collection. "Number is in reality a composite of some of the preceding operations (classes and relations) and consequently presupposes their prior construction. 'A whole number is in effect a collection of equal units, a class whose subclasses are rendered equivalent by the suppression of their qualities. At the same time, it is an ordered series, a seriation of the relations of order. Its dual cardinal and ordinal nature thus results from a fusion of the logical systems of nesting and seriation [Piaget, 1967, p. 53].'"

In summary, Piaget suggests a close developmental interdependence between cardination and ordination in the psychological conception of natural number. Further, he believes that there is a parallel between mathematical logic and the construction of knowledge by psychological processes, and that, therefore, his formalization of natural number is logically sound and all formalizations to the contrary are incomplete. It is unclear whether Piaget predicted this construction of natural number and then, in his unique empirical style, tested his prediction, or whether he first observed children and then developed this theoretical position. In either case, a replication of his work using a strict logical interpretation of the component elements of natural number is appropriate at this time.

III

THE PSYCHOLOGY OF NUMBER

While the concept of natural number may be definable mathematically or in psychological terms, it is still not clear how a child constructs the concept and what kinds of performances signify its attainment. An operational definition of the natural number concept, in the form of a set of observable behaviors, is essential to permit the inference that a child possesses the concept given a set of predetermined criteria. To create this operational definition, component elements thought to be basic to the theoretical construction of number were specified in behavioral terms. On the basis of this analysis, a performance task battery was adapted to elicit the behaviors thought to be psychologically linked to these components. The following is a brief discussion of the translation of the three philosophical treatments of number from mathematical definitions into an epistemological question involving the use of psychological terms. Specific behavioral performances that will permit the inference that the individual has constructed the concept of natural number will be identified. Strict operational definitions of concepts and tasks will facilitate this investigation.

In the Russell-Whitehead perspective, a natural number may be defined as a superordinate class encompassing all subordinate classes containing precisely that number of elements. From this view, number is first and foremost cardinal in nature and may be understood independently of ordinal assumptions. Thus, it is concluded that number is classes of equivalent collections. An individual can demonstrate his understanding of natural number by deciding whether two collections belong to the same class. In making such a determination, the individual may ascertain each collection's numerosity and then decide whether or not to place them in the same class. But this process would involve the use of numerals and numbers as a series of symbols attached to concrete objects and would not demonstrate the more basic cardinal property of number. From the cardinal orientation, number understanding may be demonstrated by the operativity of a systematic one-to-one correspondence activity, which does not demand the use of numerals.

Two collections may be said to be similar and, therefore, may be grouped under the same class, when there is an established one-to-one relationship which correlates the items of one class with the items of the other. In more specific terms, a one-to-one relationship is said to exist between two collections, X, Y, if for each item X_1 there is an item Y_1 such that no other item X_n has the same relation to Y_1 and conversely, for each item Y_1 there is an item X_1 such that no other item Y_n has the same relation to X_1.

Two classes that are in one-to-one correspondence must contain the same number of objects, but it is not necessary to count the collections to reach this conclusion. The following examples are presented to further

explain this point. When a squad of soldiers, each carrying a gun, passes some point, there are just as many guns as soldiers. A one-to-one correspondence exists between them. While the relationship between sons of the same parents and their father may be many-to-one, there is a one-to-one correspondence between an eldest son and his father.

It may be inferred from the Russell-Whitehead thesis that the construction of natural number in human conceptualization may be demonstrated by elaborating equivalent collections as belonging to the same superordinate class by (one-to-one) correspondence. If this procedure of establishing a relationship between items of one collection and items of another reveals that the collections are examples of the same class then they are instances of the same number.

Russell (1917) presents an example in which the elements contained in the collections of the months of the year, Napoleon's marshals, the twelve apostles, and the signs of the zodiac are made to correspond in a one-to-one fashion. Once the items of each collection are established in such a relationship, the collections are defined as equivalent classes and of the same number. In a later section of this paper, Piaget refutes this argument on the grounds that both properties of class and relations are employed in the establishment of one-to-one correspondence in the case presented.

For the purposes of this study the notion of class has been subdivided into two aspects of classification activity (see Piaget, 1952). The first is the qualitative aspect of classificatory logic (class inclusion). This refers to the defining property unique to all members of a given class and serves to distinguish that class category from all other class categories. An assessment situation which demands the use of qualitative classificatory logic is one in which the individual is asked to sort a number of stimuli differing on several intensional attributes such as shape, color, and the number of discrete items on each stimulus card. Another assessment situation demands the formation of appropriate classes by exclusive criteria and the subsequent comparison of the relative magnitude of class membership. This is the quantitative aspect of classificatory logic (extension). An assessment paradigm in which the individual is required to form and compare class exemplars in a stimulus array which contains class intersections and hierarchies of classes and subclasses satisfies the formal considerations of the logicist's perspective.

The ordinal position of Peano claims that the property of order is inherent in the concept of natural number. Number is thought of as an ordered series of symbols, each gaining its meaning by its relative position in the series. A value is assigned to each number according to its numerosity and its quantitative relationship to the distinct elements of the number series. But this value is not the essence of natural number. in Peano's theory. Most essential to the ordinal interpretation of number is the relationship between the elements which gives rise to their order: in any collection of two terms one must precede and the other follow. Thus, the most primitive premise in the understanding of number is the notion of a series of successors forming a progression. The progression of successors need not be composed of items or number as a string of symbols or sounds; it may be composed of points in space or moments in time.

A psychological analysis of number implies a cognitive operation demonstrating transitive inference. In this investigation, the understanding of order was assessed by a serial ordering task. The individual was required to order a number of stimuli which differed by a specified amount on the quantitative dimension of length. Further assessment required the individual to impose order on the stimuli and to locate ordinal positions on the series.

The Piagetian perspective would not argue with the notions of classes and relations set forth by these philosophical treatments of natural number. Piaget asserts that, while number may be considered an aggregate of single classes of equivalent groupings of collections, it must simultaneously be considered a series of entities ordered according to the relationship between these entities. The logical premises of number may not be made to correspond to a simple composition of classes only or to a simple serial composition only, but to both together. In human conceptualization, natural number is the operativity and simultaneously the coordination of both ordinal and cardinal notions that synthesize into a single system of quantity.

Piaget sought to define operationally the natural number concept in behavioral terms. In attempting to establish a minimum criterion for demonstrating acquisition of the number concept he rejected verbal responses indicating knowledge of the names of the entities within the number system. He similarly rejected ability to deal with the processes involved in arithmetic as an indicator of understanding of the number concept--simple addition and subtraction of numbers, as well as other manipulations of them (counting, equating sets by number), can be carried out entirely by rote. Through memorization, the child can count or compute simple arithmetic tasks without any notion of the underlying concepts of natural number.

For Piaget existence of the principle of identity (i.e., a set of elements remains unchanged irrespective of changes occurring in the relationship between the elements) is a necessary condition for all rational activity. A number is only intelligible if it remains identical with itself whatever the distribution of the units of which it is composed. In Piaget's view an individual should be able to equalize two small collections (5-7 elements) by establishing bi-univocal (one-to-one) correspondence between the terms of each collection as a minimal condition for demonstrating understanding of the natural number concept. As a demonstration of number conservation, the individual must recognize that two collections are still equal if, without adding or taking away any elements, the spatial arrangement of one of the collections is modified, so that its elements are no longer directly opposite those of the other. In a conservation of number problem, two rows of objects are arranged so that the one-to-one correspondence between their elements can readily be perceived. Then one of the rows is rearranged, by either extending or compressing the distance between its elements, so that the one-to-one correspondence is no longer perceptually given.

Thus, one-to-one correspondence seems to be the basis of the natural number concept for Piaget. However, this should not be taken as support of the Russell-Whitehead perspective even though it espoused the same behavioral demonstration as evidence of the natural number concept. Piaget differs quite clearly in his interpretation of what one-to-one

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correspondence signifies in terms of the mathematical ingredients of classes and relations and their role in the establishment of one-to-one relationships between elements of collections.

There are two types of one-to-one correspondence. It is crucial to present both to see if the Russell-Whitehead perspective utilized the procedure correctly. First, there is one-to-one correspondence based on the qualities of the elements to be considered. That is, an element of one class is made to correspond to a specific element of another class because of some quality that the two classes have in common. Piaget presents an example to show one-to-one correspondence based on the notion of class. Consider five people, five trees, and five apples made of paper; within each group each member is one of five colors. "The qualitative one-to-one correspondence would consist of putting the red person in correspondence with the red tree and the red apple, the green person in correspondence with the green tree and the green apple etc. [Piaget, 1970a, p. 36]." Piaget argues that the Russell-Whitehead thesis does not use one-to-one correspondence based on qualities for equating classes. In their example of one-to-one correspondence, Russell and Whitehead are basing natural number not on classification operations only, as they had intended, but on the notion of "units." This is the other type of correspondence. The notion of units is not based on the qualities of individual elements; it demands instead a coordinated concept involving the synthesis of classification and relational abilities.

In Russell's example of equivalent classes,

> there are no qualities of the individual members that lead to a specific correspondence between one element of one class and one element of another. We cannot say, for instance, that St. Peter corresponds to the month of January, which corresponds to Marshal Ney, who corresponds to Cancer. When we say that these four groups correspond to one another, we are using one-to-one correspondence in the sense that any element can be made to correspond to any other element. Each element counts as one, and its particular qualities have no importance. Each element becomes simply a unity, an arithmetic unity [Piaget, 1970a, p. 36].

This correspondence between the elements of equivalent collections does not imply or involve an a priori primacy of the cardinal over the ordinal.

Underlying the number concept is the need to establish similarity between equivalent collections. This involves the coordinated skills of ordering and classifying the elements of such collections in the form of one-to-one correspondence. The members of a particular collection are considered similar; they are therefore a class of like items (i.e., Napoleon's Marshals). For the correspondence to be exact (each element of the collection aligned once and only once with a counterpart in another collection), while the unique qualities of individual elements are disregarded (abstracted to form the class originally), the individual elements must be ordered. Once the order of the objects is specified, the particular arrangement of their correspondence is unimportant. If one-to-one correspondence is satisfied, then the collections may be considered subordinates of the same class, or number.

Thus a true concept of natural number implies an understanding of the notion of unit. Unit denotes the union implied in classifying an element with and differentiating that element from all other elements in a collection. Implicit in the concept of natural number is the principle that an element must be considered a unit--interrelated with other elements of the collection yet existing independently of the collection. "Only when the child can grasp that an element can at once be alike and different from another can he arrive at a true unit, hence number, concept [Elkind, in Piaget, 1967, p. 72]."

From Piaget's position the following argument can be made against the ordinal basis of natural number: Ordinals have no independent existence or meaning without the concrete quantitative value ascribed to them by the concept of natural number. The number 2 may refer in the ordinal sense to all instances of the second position in an ordered progression, but it is also a concrete value in the cardinal sense denoting some specific instance of 2 discrete objects in the real world. The number 2 refers simultaneously to both the ordinal position and cardinal value attached to it.

Piaget contends that the behavioral analog for a true understanding of natural number is demonstrated by a task of conservation of number which involves the invariance of a nonqualitative one-to-one correspondence. This understanding is in parallel development to and is a product of an undissociable synthesis of classes and relations. The development of the natural number concept in human cognition is attained when operativity of these mental abilities is realized.

A second task complementing Piaget's number conservation task was administered. Adopted from the investigations of Brainerd (1973a), this task assessed the individual's concept of unit by demanding for its proper solution a mental mechanism of nonqualitative correspondence. Brainerd had employed this task to demonstrate the cardinal properties of number understanding. In view of controversy surrounding the use of correspondence tasks for such purposes, in this investigation both paradigms are examined.

IV

PIAGET'S CONCEPTION OF MIDDLE-CHILDHOOD INTELLIGENCE

The descriptions of the basic processes underlying the concepts of classes and relations are embodied in Piaget's formalization of the logical grouping. He classified and defined by logical properties the different operational structures within these concepts that are necessary to account for the development of natural number intelligence. His formalization is characterized by appendages that are necessary for a theory but not readily perceptible in human cognitive behavior. Piaget's logico-mathematical model does not suggest in any explicit sense that the child understands the logic involved in his conceptualizations. The model was designed to characterize and describe the processes underlying the child's consciousness. Piaget attempted "to capture the essence of the child's activities and to identify the processes underlying them. The grouping is Piaget's way of describing these processes in a clear way. Therefore, the grouping is not simply a protocol listing everything that the child does. It is instead an abstraction which describes basic processes [Ginsburg & Opper, 1969, p. 132]."

The notion of comparing structure to mental processes held tremendous appeal for Piaget. When the idea of structure is applied to mental phenomena, an integrated and comprehensive picture of cognitive functioning emerges. Structure may explain the lawful development of cognitive behavior and, in turn, lead to testable hypotheses about the interrelationships between cognitive entities. Cognitive development, then, is understood in terms of an organization of elements, progressively adjusting and evolving in a predictable direction according to laws of organization, transformation, and reorganization.

> We may say that a structure is a system of transformations. Inasmuch as it is a system and not a mere collection of elements and their properties, these transformations involve laws. The structure is preserved or enriched by the interplay of its transformation laws, which never yield results external to the system nor employ elements that are external to it [Piaget, 1970b, p. 5].

Consistently applying the mathematical analysis results in cognitive operations being viewed as organized entities bound into a close-knit totality by laws governing their relations and transformation. Cognitive behavior is governed by "laws independent of the nature of the objects to which these actions are applied [Piaget, 1966, p. 171]." Cognitive elements.

are abstractions, logical consequences of the conditions determining their formation. The laws that govern the transformation of the cognitive elements are particular to the cognitive system and they exhibit the same successive forms regardless of particular external content. While the cognitive element is an entity, it gains meaning only when taken in relation to the total structure and its association with the other cognitive elements. The successive forms of cognition may be viewed in structural terms as a self-regulating closed system. The emergent totality is a new structure "which does not annex" the earlier structural form; "if anything, we have a confederation, so that the laws of the substructure are not altered but conserved and the intervening change is an enrichment rather than impoverishment [Piaget, 1970b, p. 14]."

In effect, mental development is characterized by a series of reconstructions with each successive form going beyond its predecessor. "The earliest structures observed in the coordination of actions (at the sensori-motor level) are undoubtedly derived from the structures laid down in the nervous system [Piaget, 1966, p. 234]." At a later stage of cognitive development, mental operations are abstracted from actions, performed on the physical world, that are interiorized, reversible, and characterized by the logic of structural totality (the laws relating to group, lattice, and topology). The successive forms of mental activity leading to full operativity are a function of the structure and are acquired through an interplay between experience, involving abstractions derived from actions, and equilibrium, an internal mechanism balancing the present mental state with that of the new experience.

Based on the mechanism of "reflective abstraction" originating from the individual's actions, certain connections and relationships beyond the specific properties of the particular environmental objects are "drawn out," "projected upon," and used to elaborate a new plane of thought.

> In fact, as opposed to empirical abstraction, which consists merely of deriving the common characteristics from a class of objects (by a combination of abstraction and simple generalization), reflective abstraction consists in deriving from a system of actions or operations at a lower level, certain characteristics whose reflections upon actions or operations of a higher level it guarantees; for it is only possible to be conscious of the processes of an earlier construction through a reconstruction on a new plane. This fact is not particular to scientific thought, and it already characterizes the whole development of intelligence during the transition from a hierarchical stage to one following it. In short, reflective abstraction proceeds by reconstructions which transcend, whilst integrating, previous constructions [Piaget, 1966, p. 189].

The process of reflective abstraction and the subsequent construction of logico-mathematical knowledge give rise to a certain number of elementary operational systems which then allow the substitution of deductive reasoning for direct experience.

Cognitive development is characterized by an uninterrupted series of progressive abstractions originating not from empirical objects (perception, etc.) but from the actions and operations which the individual performs

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on these objects. The process of reflective abstraction and its constructs in operational thought are embodied in a structural framework closely akin to the structure that dictates the conceptualization of mathematical thought. It may be generalized that there is a universal structure governing the functional relations of all living organizations to the environment whether biological, physical, or mental and deriving its source "from the most primitive organism-environmental matrix [Furth, 1969, p. 65]."

It is in the Bourbaki program, a system encompassing mathematical elements of every variety and characterized by laws of transformation by reflective abstraction, that Piaget finds "three not further reducible 'sources' of all other structures [Piaget, 1970b, p. 24]." From these sources, the nine logical groupings that constitute the nature of operational thought may be derived.

The first of these sources is the algebraic group. The group is characterized by composition operations (including associativity), guaranteed permanence of general identity, and reversibility in the form of inversion or negation. The next source is the lattice or network structure which denotes the relationship between the cognitive elements. This structure, by imposing a "predecessor/successor relation" on the elements, gives order to the cognitive system. The defining property of this order structure is that reversibility takes the form, not of inversion, as in the group, but of reciprocity. "Reciprocity entails not the outright elimination or negation of a factor but its neutralization, that is, holding its effect constant in some way while a second factor is being varied [Flavell, 1963, p. 209]." Finally, the third source is topology which deals with the concept of representational space, in terms of neighborhood, limits, and continuity. When combined with the other structures, this structure gives rise to more complex spatial structures of operational thought, such as measurement.

From these three sources all other structures could be derived either by the process of differentiation (limiting their generality) or by the process of combination (the adjusted intersection of combined or multiplied structures). It is the nature of these structures and their interplay that permits innumerable combinations without the restriction of contiguous additivity. The actual formulations of these structures can be viewed as analogous to the development of operational thought. The nature and development of human intellectual behavior may well be understood through the study of mathematical structures. On a more general level, it is these mathematical structures and their interpatterning that for Piaget best approximate the essential organizational structure and functions common to all living organisms.

These three structures comprise nine distinct systems (groupings). Four groupings denote operations dealing with classes and four others operations dealing with relations. The ninth is unidentified as yet. The first four demonstrate the manipulations of the union and intersection of hierarchical classes and their inverses; the latter four demonstrate the manipulations (addition and multiplication) of equivalence and difference relations of elements and their reciprocals.

Piaget contends that the structure of thought in the middle childhood period is analogous to these mathematical considerations.

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> This structure is found in eight distinct systems, all represented at different degrees of completion in the behavior of children of 7-8 to 10-12 years of age, and differentiated according to whether it is a question of classes or relations, additive or multiplicative classifications, and symmetrical (or bi-univocal) or asymmetrical (co-univocal) correspondences [Piaget, 1966, p. 179].

		Classes	Relations
Additives	asymmetrical	I	V
	symmetrical	II	VI
Multiplicatives	co-univocal	IV	VIII
	bi-univocal	III	VII

The first logical grouping (Grouping I) is the composition or union of simple class hierarchies. Emphasis is placed on the relationship between subclasses and their superordinate class. Grouping II is the combination or union of secondary classes within a class hierarchy. The union is complementary to the primary addition of classes in Grouping I. Grouping III is the intersection of subordinate classes where each component subclass of the first set is placed in multiplicative correspondence or association with each component subclass of the second. Grouping IV is the intersection or multiplication of nested subclasses in which one element of one class is set in correspondence with several elements of each of the other classes. Grouping V is the logical addition of ordered elements to form a series. Grouping VI involves the additive composition of non-ordered relations between equivalent elements. Grouping VII involves the one-to-one multiplication of ordered series. Grouping VIII denotes multiplication of symmetrical by asymmetrical relations.

In terms of cognitive activity, all possible mental operations within the middle childhood period are accounted for with this formalization enumerating the properties and characteristics common to all such sets of operations in general mathematical constructions.

> Piaget has simply not found any important way of manipulating classes and relations not caught by one or another of the nine groupings. This general approach is distinctly logical rather than empirical; in itself it says nothing whatever about whether children, in fact, think this way. What it seems to say is that if a person fully grasps the basic nature of classes and relations and the possible operations one can perform upon them, then one can reasonably impute to him cognitive structures which approach, as ideal patterns, the nine groupings [Flavell, 1963, p. 188].

Flavell argues that Piaget's "structural model-for-cognition approach resolves itself into a general context or behavior, rather than a precise and detailed statement of mode-to-behavior isomorphism [Flavell, 1963, p. 188]." These groupings were constructed to be logically possible and

complete as structures of cognition. But there has yet been no accepted empirical discovery of the existence of two of them (Grouping IV and Grouping VIII) in human cognitive behavior. In Piaget's investigation, he did not seek to assess the child's facility of thought on all the properties of the groupings. He concentrated his efforts mostly on composition and reversibility within six logical operations. This emphasis could be explained by his belief that these are the core properties of cognition, the ones from which all other properties are derived.

PIAGET'S CLINICAL METHOD

Piaget's experimental method of assessing the mental structures of the young child reflect his conception of the child of this age period. Initially, he attempted to minimize reliance on language. The child might not understand everything said to him and, even if he did understand, it might be that he could not adequately express in words the full extent of his knowledge. Thus, Piaget's questions referred to concrete events or objects which the child had before him. Secondly, understanding that the child develops concepts through an active process of manipulating the objects in his surroundings, Piaget made great efforts to devise tasks which let the child express the process and product of his reasoning by manipulating concrete objects. Primarily Piaget's data consisted of what the child did with the objects and not what he said about them.

As stated earlier, Piaget concentrated on establishing the presence or absence of six of the groupings. However, the child's understanding of such abstract concepts is not easily shown by the simple manipulation of concrete material. The range of behavioral analogues to the underlying mental structures is global and "the apparent assumption in this approach, although not explicitly stated by Piaget, is that where reasonable evidence for one or two components is found, the existence of grouping structure as a whole can be inferred [Flavell, 1963, p. 190]." This assumption has attracted intensive scrutiny. The question is, what, in behavioral terms, are the qualifications for evidence indicating the existence of one or another of the groupings (see Brainerd, 1972)?

V

REVIEW OF THE EXPERIMENTAL LITERATURE

The psychological literature abounds with studies on children's acquisition of cognitive skills. While the compilation of investigations reviewed in this section is in no way exhaustive, it does present the major sources of theoretical and empirical writings dealing with the natural number concept.

PIAGET'S STUDIES

While a considerable number of studies have been published on number concept development, only a small percentage of them have provided empirical evidence concerning the developmental relationship between the natural number concept and its elements of classes and relations. Piaget provides some insight into the ontogenesis and interdependence of these cognitive competencies in his book, The Child's Conception of Number (1952). Piaget's original studies (carried out in the 1930's and first published in English in 1952) describe broad developmental patterns relating to the acquisition of the natural number concept. The results of his in-depth investigations provide sufficient evidence to support the hypothesis of simultaneous construction of the structures of classes, relations, and number. Piaget is quite satisfied with providing a global picture of the development of operational intelligence and is not concerned with a precise psychometric approach to the phenomenon.

Using a task battery containing measures of classification, class inclusion, multiple class membership, one-to-one correspondence, seriation, and serial correspondence, Piaget discerned three stages in the acquisition of classification, seriation, and conservation concepts. Thoughts relating to each concept area develop in the same stage-like fashion and in close synchrony with each other. At the first level, the child has only a general, global impression of quantity. Piaget finds intellectual behavior at this stage consisting of simple classification and relational abilities in which the child compares, distinguishes, and orders objects on a perceptual level. The child lacks an overall plan or logical procedure when dealing with situations demanding cognitive manipulations of either the relations between elements or the dimensions characterizing objects. The child's thought is masked by centration in which he attends only to a limited amount of the available information. At this stage of operational development the child has difficulty in differentiating and coordinating the ordinal and cardinal aspects of number. His judgments, rather than being related to the foundations of number, are based on some irrelevant cues such as the length or density (dispersion) of the object array.

The second stage child has a partially differentiated, relatively precise impression of quantity which Piaget labels as an intuitive conception. The child's responses to number, class, and relational tasks do not demonstrate full operativity of the respective concepts but are influenced by the intensional (relations of similarity) properties of the stimulus set. Thus, on a conservation of number task the child may choose one-to-one correspondence as a device for establishing the equivalence of two sets of chips--provided that the elements of the model row remain opposite the elements of the row which copies it (spatial correspondence). In stage II, the construction of collections is no longer figural and the child's classification schemes show divisions and overlapping of classes. However, these overlappings are constructed by trial and error and lack the extensional quantification which is necessary for an understanding of class hierarchy (such that if A and A' are included in B, there are more elements in B than in A or in A'). The child's performance on the construction of serial orderings is usually only approximate and even if it is entirely correct, he still has required a good deal of trial and error. It is not until stage III that the child is able to appropriately partition stimulus dimensions and object arrays to discern the interplay of relationships and consider the individual element in the context of its dual class membership (extensivity). Thought based on the coordination of extension and intension Piaget speaks of as an abstract conception.

The coordination of judgments leads the stage III child to compare and contrast objects on unit terms, thus compensating for apparent perceptual inconsistencies by the logical multiplication of their intensional and extensional properties. In this way, the child is able to partition his view of objects and consider them on a multi-dimensional level, allowing for the division of a given quantity into units that are recognized as equal and yet distinct. It is not until stage III that full operativity of classes, relations, and number (characterized by the notion of unit) is realized. It is at this level of development that the child (1) guarantees the permanence of equivalent collections by one-to-one correspondence despite perceptual displacement, (2) considers a subordinate class simultaneously an entity while it remains a subportion of a larger superordinate class, and (3) views an element of a series as simultaneously larger than one element and smaller than another without perceptual validation.

Piaget posits and validates a three stage model for the development of number and cardinal and ordinal competencies, each developing through the same stages and emerging in close synchrony. It has been argued (Brainerd, 1973a; Flavell, 1970) that the methodology by which Piaget collected and analyzed his data is inadequate because the emerging pattern of each ability was based on comparisons between separate subject groups. Measuring each ability in a different group of subjects, he compared the average age of emergence. He made no effort to make within subject comparisons on a longitudinal basis to get a more accurate picture of the synchronic patterns of these developmental phenomena. Furthermore, there is some question about the appropriateness of his operational definitions and the employment of certain tasks as analogues to the mathematical formulations of number. Although the field is and will be eternally grateful for Piaget's groundwork in this area, what is now needed is further logical analysis based on mathematical considerations, appropriate operational

definitions (in behavioral terms), and new psychometric techniques for analyzing developmental data.

In the non-Genevan research, reviewed in the following discussion, while the methodological procedures employed are more psychometrically sound, the question of operational definitions of the concepts under study remains unresolved. Resolution of this issue has not been forthcoming because, for the most part, Piaget's original conceptual measures have only been adopted and re-employed.

VALIDATION STUDIES

There have been several validation studies of Piaget's number research (i.e., Dodwell, 1960, 1961; Elkind, 1961; Wohlwill, 1960; Wohlwill, Devoe, & Furaso, 1971). Their findings suggest that a child between the ages of four and ten displays characteristic performance patterns when confronted with problems requiring reasoning or logical manipulations of concrete objects that involve number concepts (Elkind, 1961). Initially the child's cognitive behavior may be stimulus-bound, his judgments subject to the influences of the perceptual presence of certain configurations. During this period the child develops the abilities that allow him to conceptually organize his environment. His adaptation to reality then becomes more natural in the adult sense.

In general, these studies tended to support Piaget's findings regarding the sequence of conceptual levels that lead to the operativity of natural number understanding. More importantly, to date, only a limited number of studies have dealt with the relationship among abilities or structures of the natural number concept in the middle childhood period. The conclusions to be drawn from these examinations are not too clear.

Dodwell's Studies

Dodwell (1960, 1961) administered a battery of Piagetian tasks to a large sample of kindergarten and first and second-grade Canadian children. He was concerned with the child's performance on conservation, cardination (qualitative and non-qualitative correspondence), and ordination (seriation and cardinal-ordinal correspondence). He reported that there was no sequential pattern distinguishing cardinal and ordinal abilities. Further, he was able to identify the three major levels of conceptual thought in the development of number conservation. Using scalogram analysis, he found the expected sequential dependence supporting the hypothesis that an operational grasp of natural number implies a prior attainment of class and relations competency. Dodwell found that children who solve ordinal-cardinal problems (quantitative extension) can also solve seriation and one-to-one correspondence tasks, but generally not the converse.

In Dodwell's 1960 study, several children were able to respond properly to tasks demanding cardinal and ordinal coordination before they could deal with either classes or relations separately. It was also found that for several children it was not necessarily true that if they could deal with classes and relations separately they could deal with number tasks which involve both. Equally as important is the finding that there is not a great deal of regularity in this pattern.

In a follow-up study, Dodwell (1962) investigated the extent to which the young child, in developing the concept of number and conservation of physical quantities, also develops the concept of "class of objects." The mental operations and linguistic skills necessary for the child to deal with the elementary logic of classification and the composition of classes were assessed using a procedure similar to that developed by Piaget. Dodwell attempted to minimize the ambiguity of task questions. Three types of responses were obtained on an array of class inclusion questions. The response categories were similar to the stage performance observed in the development of the number concept found in both his and Piaget's earlier investigations. Correlational treatment of his data did not reveal a clear relationship between the development of the concept of number and the development of the concept of classes, although both concepts develop within the same age range. Dodwell aptly concludes his overall evaluation of Piaget's research on natural number concepts with the following statement: "While Piaget is on the whole correct in his description of the child's understanding of number, the pattern of development is neither as neat, nor as rigid, as he would have us believe [Dodwell, 1961, p. 35]."

Other Investigators

Wohlwill (1960) designed a series of tasks paralleling Piaget's number experiments by translating the original verbal technique into a nonverbal format. The results of his study confirm the existence of a relatively uniform developmental sequence in the area of number concept, as suggested in Piaget's theoretical position. While Lovell and Ogilvie (1960) provide support for the three stage model of cognitive development proposed by Piaget, they also note that the stages are not clear-cut.

Slightly more affirmative testimony to parallelism between conservation, classification, and seriation is given by Almy, Chittenden, and Miller (1966). They report some indication that the performances on a class task (involving collections of floatable objects) and a task of serial ordering (staircase) follow a trend similar to that found among a number of conservation tasks.

BRAINERD'S STUDIES

Brainerd (1973a, 1974) provides experimental evidence in support of the ordinal theory of natural number development. He finds that "'number development' is synonymous with the emergence of children's grasp of natural number in ordinal theory" as formalized by Peano (Brainerd, 1974, p. 3). In an initial investigation, Brainerd developed an alternative task format (based on Piaget's Groupement model) for uncovering the cognitive competencies within the middle childhood period. He found "that all four relational groupements appear much earlier in life than the four class groupements [1972, p. 12]." Later, in task settings which were operationally derived from mathematical philosophy, he again found the emergence of relational concepts to precede the emergence of class concepts (1973a). Results from his investigations (1973b, 1974) of the natural number concept have led Brainerd to conclude that ordinal understanding precedes the use of number and cardinal understanding follows some time after number use.

This sequential pattern was further substantiated in a transfer of training experiment (Brainerd, 1973c).

> Explicitly, number development is viewed as a three phase process: First, during late preschool and kindergarten in most Western children, ordination skills (particularly quantification of transitive-asymmetrical relations) begin to appear; second, during the first two or three elementary school years in most Western children, natural number skills (particularly arithmetic manipulations of the first few positive integers) begin to appear; third, during the third and fourth grades in most Western children, cardination skills (particularly quantification of classes via correspondence of elements) begin to appear [Brainerd, 1974, pp. 3-4].

Number development is found to be a continuous process, a clear developmental progression of ordinal understanding leading to natural number ability and followed by cardinal comprehension. Brainerd's results on three normative studies and a training experiment provide support for this sequence. These findings are in obvious discord with Piaget's theory which predicts that the three concepts of classes, relations, and number emerge together.

In general, with Brainerd's contributions as the exceptions, the findings of investigators differ from Piaget's discoveries less in terms of the developmental sequences identified, than in the specific age norms for particular stages. Intra-level variability between hypothesized corresponding number-related abilities have been discovered as well as limited inter-situational and inter-task generalizability (additional discussion of this phenomenon may be found in: Braine, 1968; Bruner, Olver, Greenfield et al., 1966; Elkind, 1961; Lunzer, 1960).

SUMMARY

The bulk of the replication studies support the notion that the child's ability to conserve quantity and comprehend related concepts of classes and relations is arrived at in an orderly progression in the manner described by Piaget. But more important is the question of the relationship among the natural number abilities as a synchronous emergent pattern. Does the concept of unit parallel the emerging capacity to deal simultaneously with objects on the basis of their likenesses and differences? A number of researchers have attempted to validate the belief that the abilities Piaget attributed to a given stage correlate as closely as he suggested. But the strict empirical evidence for such tight synchrony may be confounded by a number of factors. First, the synthesis of these abilities at any one level represents the attainment of one stage and the starting point of the next. Thus, children in a normative cross-sectional investigation may include a wide degree of attainment levels depending on the age range studied. Second, it has been discovered that abilities during the period of their formation are not applied equally to all settings (see Hooper, Goldman, Storck, & Burke, 1971, for a review of these

studies). These considerations make it imperative that multiple measurement and long-term assessment techniques be employed in cognitive developmental research.

VI

EXPERIMENTAL DESIGN

The treatises of Peano, Russell and Whitehead, and Piaget form the theoretical basis and provide the philosophical definitions for studying the nature and development of the natural number concept in human conceptualization. Underlying the ordinal, cardinal, and ordinal-cardinal perspectives are primitive assumptions which were employed in the development of task materials and procedures in the following experimental investigation. While this study may shed some light on the adequacy of either the ordinal or cardinal philosophical treatments in accounting for the knowing of natural number, the principal interest of this examination is Piaget's empirical model of cognitive development.

Behavior patterns of children five and eight years of age in response to number-related tasks were examined and compared to Piaget's model in terms of acquisition levels underlying theorized cognitive competencies employed to achieve task solutions. Evidence to support Piaget's contention that the natural number concept is the constructive synthesis of conceptual elements of classes and relations is provided if the emergence of these concepts reveals a synchronous developmental acquisition pattern.

An effort was made to ensure that the three distinct concept task types were of equal assessment sensitivity by employing a three level acquisition analysis and standardized scoring criteria across concept areas. An hypothesis relating to the equivalence of inter-concept degree of difficulty was also tested.

Concept acquisition was measured by two complementary assessment situations, as suggested by Tuddenham (1971) and Osherson (1974). The first situation sought to provide understanding of the concept by simple demonstration, while the second situation required the child to spontaneously employ the concept to override perceptual impressions. The consistency of the subject sample's performance patterns on each pair of concept tasks and the degree of association between the tasks were also examined. Differential responses to within concept tasks would justify the weighting of these performance measures to assign a concept acquisition stage to an individual.

Because of Brainerd's extensive research effort in the area of cognitive developmental processes and his results which contradicted Piaget's model, an additional analysis was performed employing a set of behavioral indices similar to those he chose. The task formats used were analogous to those employed in Brainerd's investigations. A scoring procedure was developed to assess response types characteristic of each of three stages of cognitive performance for each concept area. Statistical analysis was of the same form performed on the main task battery. An effort was made to relate the performance patterns of the subject sample on both (main and Brainerd) sets of conceptual measures.

SUBJECT SAMPLE

Subjects for this study were 60 kindergarteners and 60 third graders drawn randomly from class lists of four public elementary schools in the city of Beloit, Wisconsin. The schools were selected because they served a cross section of the city's diverse racial, ethnic, and economic population of 40,000 people. The distribution of the subject population by age and sex is described in Figure 1.

Kindergarten	Third Grade
Total N = 60	Total N = 60
30 boys	30 boys
30 girls	30 girls
Mean Age = 69.7 months	Mean Age = 106.6 months
Age Range = 63.6-76.8 months	Age Range = 99.6-117.6 months

Figure 1. Distribution of the subject population by age and sex.

GENERAL ADMINISTRATION PROCEDURES

The present study was a part of a larger longitudinal investigation (Hooper & Klausmeier, 1973) involving a comprehensive array of Piagetian tasks and a series of measures based on the Conceptual Learning and Development Model developed at the Wisconsin Research and Development Center for Cognitive Learning. The tasks in this experiment were administered in the first and last sessions of a three session task battery, each of which was approximately twenty minutes long. The order of presentation of the first session was counterbalanced by a Latin Square procedure, while the order of the second and third session presentations was fixed except for the counterbalancing of Class Inclusion A. It was felt that the order of the latter sessions would not facilitate task performance. The tasks of primary concern to this study are the assessment situations from Session I and the class inclusion and ordinality tasks from Session III.

<u>Session I</u>
Seriation
Conservation of Number
Unit
Dichotomous Sorting
Arithmetic Proficiency

<u>Session II</u>
Memory
Groupement Tasks
Combinatorial Reasoning
Some-All

<u>Session III</u>
Conservation of Weight, Length
Ordinality of Weight, Length
Class Inclusion A
Class Inclusion B

TASK PROCEDURES

Assessment situations were adapted as those being the behavioral isomorphs to the formal definition of number and its partitioned elements of classes and relations. By administering two tasks for each concept, adequate information was provided to infer a child's understanding of number-related concepts. A considerable number of questions were asked of each subject for each task in order to provide a sufficient amount of data with which to classify the quality of the subject's response pattern across concepts. The initial task within each concept area assessed the essential understanding of the concept in question. The second task sought assessment of appropriate use of the concept in strategies to solve problems. Task one, therefore, would be more easily mastered than task two in each concept area. The experiment attempting to replicate the investigations of Brainerd included measures of transitive inference, class understanding, and arithmetic proficiency. The construction of these assessment situations was a direct product of task descriptions presented by Brainerd in his number monograph (1973a). Task protocols for all tasks are presented in Appendix A.

Relations

Relations can be interpreted in the behavioral sense as the ability to perceive and establish relationships between elements and employ those relationships to (1) establish a series, (2) perceive magnitude in relative terms, (3) impose order and locate positions within a series, and (4) relate items in one series to other elements in a second series in terms of their ordered positions. Responses to seriation and serial correspondence tasks were used to provide evidence that the child had developed the cognitive apparatus germane to relationality.

Seriation. Seriation measured the child's ability to perceive the asymmetrical transitive relations between elements of varying quantity. The task required the subject to impose order on an array of stimuli and then systematically construct an ordered progression. Serial correspondence assessed the child's ability to employ order to locate an element in one series and to transpose that location to a position and correlate in a second series.

The seriation task began with a warm-up trial involving the ordering of four sticks (strips of laminated cardboard 1-centimeter wide). The sticks increased in length by 1-centimeter increments from an initial length of 2-1/2 centimeters. The child was then asked to order an array of seven sticks. The experimenter noted help requested, intermediate orders, and the final stimulus configuration. The flexibility of the child's relational thought was assessed by asking him first to predict where in the ordered array an additional three elements would be placed and then to make the actual insertions.

Correspondence. In the next phase of testing, the ten sticks ordered previously remained while the experimenter introduced ten circles increasing in diameter by increments of 1.3 centimeters from 3 to 14-1/2 centimeters. The subject was requested to order the circles in the same direction as the sticks and then match each stick to the circle that it would go with

(ordinal correspondence). Once this was achieved the experimenter extended the stick array so that the circles were no longer opposite the corresponding sticks. The child was asked to find the circle which belonged to a particular stick pointed to by the experimenter. This procedure was repeated twice using a different stick each trial. Perceptual correspondence was then reestablished and followed by a similar procedure for the remaining compression and scrambled cases.

Classes

Sorting. Basic to the notion of classes is the construction of collections of objects founded on a similarity-of-attributes basis. An understanding of classification was assessed in a dichotomous sorting task in which the subject was asked to classify items according to one dimension and then to regroup the same items on another dimension. This task involved the sorting of twenty-two laminated index cards upon which figures were drawn (one or two squares or circles colored either red or blue). The cards were presented in a random display to the subject who was told to "divide all the drawings into two bunches." After this trial, the consistency of the sort, according to the dimensions of either number, shape, or color, was recorded by the experimenter. For the next two trials the stimuli were shuffled, reintroduced, and the child was instructed to "divide all the drawings into two bunches, but this time in a different way."

Class inclusion. Mobility of thought in terms of classification schemes was further assessed by measuring the subject's ability to manipulate hierarchical classes and class intersections. The ability to form and recombine class-subclass relations was assessed by two class inclusion tasks in which the subject was required to react in accordance with both the qualitative and quantitative properties of the items presented. These tasks demanded the operatory mental scheme of inclusion, a comprehensive understanding that each item in the perceptual field could be viewed as jointly belonging to a subclass and a larger superordinate class simultaneously. For example, the subject is asked whether a subclass is larger than the superordinate class that encompasses it ("are there more red figures or triangles?"--Class Inclusion B).

Class Inclusion A (adopted from Youniss, 1971) consisted of two stimulus presentations, cards containing drawings of either geometric figures or children. The order of presentation for these stimuli was counterbalanced--30 subjects at each grade level received the geometric figures first and the children second; the remaining subjects were administered the stimuli in the reverse order. Class Inclusion B was adapted from the investigations of Kofsky (1963) and was presented after Class Inclusion A. Class Inclusion A differed from Class Inclusion B in that when the stimuli were presented in Class Inclusion A, the subjects were asked to count both the superordinate class and subclasses. It was felt that such direction would benefit those subjects who possessed the inclusion ability but who failed to interpret the verbal instructions correctly. (For further discussion of this procedure the reader is referred to Youniss, 1971.) It was also felt that the card with the more familiar stimulus (a drawing of children) would facilitate the correct employment of the inclusion ability.

Number

Conservation of number. Number understanding is revealed when quantity is viewed as a qualitative totality structured by differing individual elements. In conception the whole is divided into units. These units, or elements, are equated to form the collection, yet are distinct due to the relationship that exists between them. Basic to this understanding is the notion that the quantity of a given collection remains invariant despite changes in that collection's perceptual arrangement. The two-phase conservation of number task employed in this investigation was adapted from Piaget's (1952) experiments. In the prediction phase, the subject responded to three hypothetical questions about the equivalence of two rows of chips if one were to change their arrangement. In the deformation phase, the subject was asked similar questions after the chips in one of the rows were spaced further apart while the chips in the other row remained the same.

Unit. The Unit task requiring the employment of a non-qualitative correspondence ability for its solution was adopted from Brainerd's investigations relating to the number concept. Brainerd used this task as an analog for Russell's correspondence mechanism employed to establish class similarity. Earlier in this presentation, it was argued that this type of correspondence used the notion of relations and, consequently, unit as well. Accordingly, it is more appropriate to use this task in the number section of the test battery than to use it as a measure of classification competency.

The Unit task required the child to judge the equivalence or non-equivalence of classes of parallel rows of dots. Two classes of red and green dots were presented in six arrangements (on six stimulus cards) varying in numerosity (6, 8, or 10 dots in each class) and density (dispersed in length over either 20 or 25 centimeters). The subject was not permitted to count the classes. In order to establish class similarity he had to establish a one-to-one correspondence relation between the elements of the two classes. "The arrangement of the two classes precludes non-correspondence solutions based on perceptual factors, e.g., when the classes are equal their elements are arranged in parallel rows of unequal length and when the classes are unequal their elements are arranged in parallel rows of equal length [Brainerd, 1973a, p. 45]."

BRAINERD'S ASSESSMENT TASKS

Relations

Ordinality. For Brainerd the most obvious behavioral counterpart for the formal definition of relations is the understanding of the asymmetrical quantitative relationship among elements of an ordered series. The necessary information for assuming the mental operation of transitive inference is provided in two assessment situations involving length and weight. Three elements differing by a small quantity on either dimension of weight (for clay balls) or length (plastic tubing) are presented to the subject. The relationships between the first two elements, i.e., A < B, and between the second two elements, B < C, are established. The subject is required to infer the asymmetrical relationship between A and C.

Classes

Cardinality. This task was employed as the unit task and is described in the number section of the main study task battery.

Number

Arithmetic proficiency. An index of number understanding was devised as essentially an achievement test of addition and subtraction facts. Arithmetic skills were evaluated using a series of equations involving the first four positive integers (16 addition facts for the kindergarten sample and 32 addition and subtraction facts for the third graders).

SCORING PROCEDURES

In order to identify each child's level of cognitive development with reference to natural number understanding, a scoring scheme was imposed on the child's responses to each concept area (classes, relations, and number) and within each concept to each component task. A scoring procedure was established a priori using the definitions of each concept and demonstrable behavioral analogues necessary to infer understanding of the concept. In keeping with Piaget's analytical method, the subject's responses to each task were categorized by the quality of the response approach employed to solve the problem.

Three levels for each task within a concept were defined and identified. The same criteria for each task were applied across tasks and to establish stage designations for each concept. The levels were defined as follows:

Level 1--responses demonstrated a fixed, inflexible response pattern that was perceptually bound to the stimulus array. Responses were in reaction to stimulus presentations and did not demonstrate that the subject was going beyond the information presented to impose a cognitive rule.

Level 2--responses indicated the emergence of a cognitive rule or device, but were not consistent. While the subject was able to demonstrate understanding of a concept in one setting, he was unable to apply this concept to all appropriate situations.

Level 3--responses demonstrated a flexible, generalizable, and consistent mental competency. The subject was able to manipulate and structure the stimulus information to solve problems involving the logic of the concept in question.

The same criteria employed to establish achievement levels for each within concept task were used to establish conceptual stage assignments. The competency of each subject by concept was recorded in three stage designations.

Relations

Seriation

Theoretical Considerations

Level 1 Subject cannot construct order successfully. Subject cannot perceive order nor the relations between elements in an order.

Level 2 Subject can construct order successfully but can only perceive order on a trial and error basis. Subject is not able to perceive relationships between items unless allowed to manipulate elements.

Level 3 Subject is able to perceive order and can construct an ordered progression successfully. Subject can judge relationships between elements and is flexible in ordering these elements according to their relative magnitudes.

Task Performance

Level 1 More than one misplaced stick (of 7) or circle (of 10)

or

fewer than two correct placements of the three additional sticks.

Level 2 All ordering of sticks and circles correct plus correct placement of the three additional sticks

or

one circle or stick out of order plus three correct placements and at least one correct prediction of the three additional sticks.

Level 3 Correct ordering of sticks and circles and three correct placements and three correct predictions of the three additional sticks.

Correspondence

Theoretical Considerations

Level 1 The subject's response tendency is perceptual in nature. The closest stick to a particular circle tends to be chosen as its analogue in that series.

Level 2 The subject's responses tend to have the mechanism of correspondence built in, but its employment is inconsistent. While the subject may use correspondence when told to by the experimenter, he does not always spontaneously employ this device when the situation warrants its use.

Level 3 The subject employs correspondence spontaneously when required by the demands of the assessment situation.

ERIC

Task Performance

Level 1 More than two incorrect alignments on the 10 stick-to-circle correspondence

or

five or more incorrect correspondence responses to the nine trials of the compression, extension, or scrambled cases.

Level 2 All correspondence correct on the stick-to-circle task plus five of nine correct responses on extension, compression, and scrambled trials

or

two incorrect correspondence on the stick-to-circle task plus six of nine correct responses on extension, compression, and scrambled trials.

Level 3 All correspondence correct on initial task plus nine of nine correct responses on extension, compression, and scrambled cases.

Relations

Theoretical Considerations

Stage I Subject is unable to see relationships and cannot impose order to construct progressions. He is unable to go beyond the perceptual impression presented by the elements of a relational task.

Stage II Although the subject could construct order and possibly make initial correspondence between two ordered arrays, he is unable to use the cue of the relative position of elements to consistently correspond elements or predict the additional placement of elements into a single order.

Stage III The subject demonstrates competence in the use of order to establish a series. He is able to perceive the relations between elements of a single array and can envision how any element can be made to correspond to a correlate in a like array of ordered elements.

Task Performance

Stage I Either more than one incorrect on the ordering of sticks or circles or two incorrect on correspondence of stick-to-circle

or

more than two incorrect placements of the three additional sticks

or

six or more incorrect responses to the nine extension, compression, and scrambled correspondence trials.

ERIC

Stage II — May have one incorrect response in each of the orderings of sticks and circles, two incorrect responses to the stick-to-circle correspondence, but must respond with at least two correct placements of the three additional sticks and four correct correspondence responses to the nine extension, compression, and scrambled trials.

Stage III — Correct responses to all seriation and correspondence problems.

Classes

Dichotomous Sorting

Theoretical Considerations

Level 1 — Subject is unable to classify elements consistently, using the same criterion.

Level 2 — Subject is able to classify elements on one dimension but uses the same criterion to reclassify the elements on additional sorts.

Level 3 — In classifying elements subject exhausts all possible classification criteria.

Task Performance

Level 1 — Zero correct sorts.

Level 2 — One or two correct sorts.

Level 3 — Three correct sorts.

Class Inclusion

Theoretical Considerations

Level 1 — Subject cannot simultaneously compare a class and its subclasses.

Level 2 — Subject can form the superordinate class and inconsistently compares the whole to its parts.

Level 3 — Subject can compare whole to parts and is able to establish their relationship. He can form the intersection of two classes and is capable of establishing its relation to either of these superordinate classes.

Task Performance

Level 1 — Less than six correct responses to the nine inclusion questions on both tasks.

Level 2 — Six to eight correct responses to the nine inclusion questions on both tasks.

Level 3 — Nine correct responses to the nine inclusion questions on both tasks.

Classes

Theoretical Considerations

Stage I	Subject is unable to form classes and subclasses and cannot consider the relations between the class whole and its parts.
Stage II	Subject can determine class membership and possesses the ability to partition collections into subparts but not on a consistent basis.
Stage III	Subject can form classes consistently and exhaustively and possesses the ability to compare class components.

Task Performance

Stage I	Not one consistent sort and less than eight of nine Inclusion responses correct or one correct sort and less than five of nine Inclusion responses correct or two or three sorts and less than four of nine Inclusion responses correct.
Stage II	At least eight correct Inclusion responses correct or one sort and at least five Inclusion responses correct or two or three sorts and at least four Inclusion responses correct.
Stage III	Three sorts and nine correct Inclusion responses.

Number

Conservation of Number

Theoretical Considerations

Level 1	The subject's response pattern indicates that he is influenced by the perceptual array or by the thought of having the elements of a class moved from their positions.
Level 2	The subject's responses indicate that he recognizes that the number of elements in a collection remains constant when told of possible deformation but is inconsistent in his responses when the deformation takes place.
Level 3	The subject recognizes that the number of elements in a collection remains the same despite change in their appearance.

Task Performance

Level 1 Less than four of six correct responses.

Level 2 Four or five correct responses.

Level 3 Six of six correct responses.

Unit

Theoretical Considerations

Level 1 The subject's responses indicate a fixation on the perceptual similarity of the two collections on some irrelevant dimension.

Level 2 The subject is aware that he has to consider not only the length of a collection's configuration but its density in determining its similarity to another collection, but he cannot consistently coordinate both dimensions simultaneously.

Level 3 The subject considers the individual elements of a collection as units and can respond appropriately to any set of stimulus configurations.

Task Performance

Level 1 Less than three of the six cards correct

or

three cards correct but two being cards B and C.[1]

Level 2 Three cards correct not including both B and C.[1]

Level 3 All six cards correct.

Number

Theoretical Considerations

Stage I The subject's responses indicate little or no conservation skills. His performance is marked by an inability to use units as a way of responding to questions demanding the use of number.

Stage II Subject has a nearly perfect response pattern to conservation but the notion of unit is not employed consistently.

Stage III Subject has total conservation ability and employs units to judge the number of elements when comparing collections.

Task Performance

Stage I Less than four conservation questions correct and less than any three unit cards correct

or

less than five conservation questions correct and three unit cards correct with two being B and C.[1]

[1]Cards B and C require only one response that demands direct employment of the unit concept.

Stage II At least four conservation responses correct and three unit cards correct not including both B and C[2] (in which case four unit cards correct)

or

at least five conservation responses correct and any three unit cards correct.

Stage III All conservation questions correct and all unit cards correct.

Brainerd's Task Array

Ordinality

Theoretical Considerations

Stage I The subject does not display a capacity for ordering. The subject accepts the perceptual equivalents of the stimuli and does not construct the ordered progression that the initial quantitative comparisons suggest.

Stage II The subject is capable of ordering objects but this capability is dependent on task material.

Stage III The subject is capable of ordering objects quantitatively regardless of task materials.

Task Performance

Stage I Less than one series of weight or length responses correct.

Stage II At least one series of weight or length responses correct.

Stage III Both series of weight and length correct.

Arithmetic Proficiency

Theoretical Considerations

Stage I An unsatisfactory arithmetic achievement score as determined by the teachers and principals of five schools participating in Brainerd's natural number study (1973a, p. 69).

Stage II A satisfactory score.

Stage III A superior arithmetic achievement score.

Task Performance

Stage I Kindergarten: five or less equations solved. Third Grade: sixteen or less equations solved.

Stage II Kindergarten: as many as six but not more than eleven equations solved.

Stage III Kindergarten: at least twelve equations solved. Third Grade: at least twenty-eight equations solved.

[2]Cards B and C require only one response that demands direct employment of the unit concept.

Cardinality

This task is scored in the same fashion as the unit task.

TESTS USED

Responses to both main study and Brainerd tasks, which comprise the assessment of the three concept areas, were categorized using the scoring system described earlier. For each grade level and the combined sample, the number of subjects assigned to each task level was tallied. The percentage of subjects at or passing each task level was computed in order to make more distinct the relative difficulty of the within concept tasks. McNemar's Test of Equality of correlated proportions was employed to assess the relative degree of difficulty between concept tasks for subjects passing each ranking. (It must be pointed out that this is a binomial test and assumes that the probability of a difference in the predicted direction is .50.) To test the strength of association between pairs of tasks, a 3 x 3 contingency table, reflecting the three levels for each task, was constructed for each grade level and the composite sample. Each cell in the table represented the frequency of joint observations at the particular task levels indicated by the row and column headings. Because the data (1) were categorical, (2) were summarized in ordinal rankings for each task, and (3) contained many ties at similar rankings, the strength of the association between tasks was analyzed using the Gamma statistic (see Goodman & Kruskal, 1954, 1959, 1963).

Gamma provides a measure of association based on the number of agreements and disagreements between the orderings of any two variables for all untied pairs of individual performances. It expresses the probability that two measures will show the same relative order in both rankings rather than a different order, and depends on the number of inversions in the order of the two variables for all pairs of individuals untied on any ranking. A simple inversion in order exists between any pair of individuals _a_ and _b_ when _a_ receives a ranking of 1 on variable x and a ranking of 3 on variable y, and _b_ receives a ranking of 2 on variable x and a 1 on variable y. When two rankings for variables x and y are identical, no inversions in order exist. Gamma is equal to the probability of obtaining the same ordering less the probability of obtaining different orderings for all untied pairs of individuals. For example, suppose that a pair of subjects were drawn at random from the 60 actually observed in each grade-level sample. Given that these subjects were not assigned equal rankings on either of the two variables, is it a better bet they show the same or different ordering on x and y? A Gamma score of .50 would indicate that it is a much better bet that an untied pair has the same ordering on the two variables, since the probability of finding a pair with the same ordering is 50 percent more than the probability of finding a pair with a different ordering between all possible untied pairs that might be drawn.

Gamma not only reflects the strength of the association between two variables but it also indicates the general tendency toward monotonicity in this relationship (Hays, 1963). Essentially, Gamma reflects the form of the relationship between two variables which may be either monotone or not related at all. In general, Gamma scores range from zero to plus or minus one with one indicating a perfect monotone association and zero indicating a high degree of independence.[3] Examples of contingency tables

[3]Note that independence implies that Gamma = 0, but Gamma = 0 does not necessarily imply independence.

that produce perfect monotone associations (Gamma = 1) are provided in a, b, and c of Figure 2. Tables with like patterns containing low frequencies in the off diagonal cells will yield a high Gamma value. Gamma is zero in Table d. As indicated by the pattern in this table, the variables are not related. Gamma computations were also employed in determining the relative difference in task performance between grade-level samples. Strong positive Gammas between age-level comparisons may be interpreted as differences related to age.

	a			b			c			d		
N = 60	20	0	0	12	12	0	12	0	0	12	0	12
	0	20	0	0	12	12	12	0	0	0	12	0
	0	0	20	0	0	12	12	12	12	12	0	12
	Gamma = 1			Gamma = 1			Gamma = 1*			Gamma = 0		

*A case of complete curvilinear association.

Figure 2. Sample tri-level contingency table.

TREATMENT OF DATA--MAIN STUDY

Principal items of interest in the treatment of data from the main study were

1. to assess the relative degree of difficulty between complementary concept tasks. Discrepancies in task difficulty in the direction hypothesized would support the assignment of stages to performance on the conceptual tasks.

2. to establish the association between intra-concept tasks. Strong association would justify the use of such tasks as measures of the same concept.

3. to assess the inter-concept degree of difficulty. Similar proportions of subjects passing at each stage for each concept area, which would indicate equivalent difficulty among the three concepts, support the contention that the task features, the cognitive demands of each measure, and the scoring criteria employed were of equal assessment sensitivity.

4. to establish the association between concepts. Indices of association would indicate the form and strength of relationship between concepts.

5. to examine within subject variability relating to the synchronous appearance of the three concept areas. This analysis would suggest either the co-emergence of the three or the possible primordial status of one of the natural number concepts.

6. to depict possible developmental patterns.

The analysis of the conceptual stage designations was conducted following the same general procedure employed for the within concept task

inquiries. Subjects were assigned to stages for the three concept areas based on their task performances. The number of subjects ranked at each stage for each of the three concepts was tabulated. To reveal the relative degree of difficulty across concept areas, for each grade level and the total sample, the proportion of subjects at or beyond each stage was computed. The emergent pattern of the three concepts was investigated by (1) constructing 3 x 3 contingency tables of pair-wise concept comparisons, (2) computing the percent of concordant pairs, i.e., the composite of equal rankings on each pair-wise comparison of concepts taken from the top left to bottom right diagonal, and (3) computing Gamma for each comparison, indicating the degree of like rankings between concepts.

The consistency of concordant rankings on the three concept areas was evaluated using the Cochran Q technique. Employed at each of the three stage designations, this procedure assesses whether the same proportion of individuals were judged at a particular stage for the three concept areas.

The suggested developmental pattern revealed by the data was tabulated by observed and expected frequencies and was schematically presented.

Consideration was also given to the presentation effects for the two orders of Class Inclusion A. A t test for independent samples was run for each grade level and for the combined subject sample. This analysis was undertaken to determine whether receiving more familiar stimuli first and neutral stimuli second had a facilitative effect on Class Inclusion responses.

The effect of age on conceptual performance was assessed by comparing attainment levels between grades for each task using the Gamma statistic. The main effects of sex were not of concern in this investigation.

TREATMENT OF DATA--SUPPLEMENTARY STUDY

Principal items of interest in the treatment of data from the supplementary study were

1. to assess the relative degree of performance difficulty among Brainerd's conceptual measures. Equivalent difficulty among the three concept measures would indicate that the task features, the cognitive demands of each measure, and the scoring criteria designed to assess the development of the natural number in human conceptual processes were of equal assessment sensitivity.
2. to establish the form and strength of the relationship between Brainerd's number concept measures by the use of the Gamma index of association.
3. to examine the within subject variability relating to the synchronous appearance of the three concept areas. Such analysis would suggest either the co-emergence of the three or the possible primordial status of one of the natural number concepts as assessed by Brainerd's measures.
4. to depict possible developmental patterns suggested by the data collected on Brainerd's measures.
5. to determine the form and the strength of the relationship between concept measures employed in the main portion of the study and those instruments developed by Brainerd.
6. to relate discovered patterns of natural number development to levels of assessment sensitivities within each study.

VII

RESULTS

THE MAIN STUDY

Initial Considerations

Initial consideration was given to evaluation of presentation effects for the two orders of Class Inclusion A. There was a notable absence of any significant order effects for the subjects who received the more familiar stimulus first (combined sample, t = 1.07; kindergarten, $\underline{t}$ = .35; third grade, $\underline{t}$ = 1.30; all $\underline{p}$ > .25). As anticipated there was a marked age effect across all concept tasks (as reflected in significantly strong Gamma scores, all above .68, $\underline{p}$ = .0001).

Intra-Concept Analysis

The general performance patterns for each task by grade level and for the overall composite sample are presented in Tables 1, 2, and 3.[4] The figures in the first set of tabulations (A) reflect the number of subjects performing at each of the three task levels. The figures in the second set of tabulations (B) indicate the percentage and number (left corner of cell) of subjects at or beyond each level on the individual tasks.

The composite sample showed that the second task within each concept area was clearly more difficult than the first. The pair-wise comparison of concept tasks shows the percentage and number of subjects attaining at least the second level of the second task to be less than those attaining at least the second level on the first task for class and number concepts (unit, 53 percent versus conservation of number, 69 percent; class inclusion, 53 percent versus sorting, 88 percent). By a binomial test these results are significant at the .001 level. Performance differences are notable at the third level for both relational and number tasks (correspondence, 15 percent versus seriation, 49 percent; unit, 13 percent versus conservation of number, 58 percent). These results are significant at the .001 level. Differences in performance between concept tasks at the second level for relations and the third level for classes are negligible and insignificant.

With one exception, this description of task performance represents the general pattern of the subject sample when it is partitioned by grade level. At the third grade level equal proportions of subjects passed number tasks at the second level.

[4]Inter-rater agreement employing the scoring procedure was 100 percent. Corrected error rate for scoring the entire sample was less than one per one hundred entries.

TABLE 1

INTRA-CONCEPT TASK COMPOSITE--TOTAL SAMPLE

A. Total number of subjects at each task level.

	Relations			Classes	Number	
	Seriation	Correspondence	Sorting	Class Inclusion	Conservation of Number	Unit
Level 1	43	44	15	56	37	57
Level 2	18	58	81	43	13	47
Level 3	59	18	24	21	70	16

B. Percent of subjects performing at or beyond each task level. Numerals in upper left corner of each cell represent the number of subjects reflected in the cell percentage.

	Relations			Classes	Number	
	Seriation	Correspondence	Sorting	Class Inclusion	Conservation of Number	Unit
Level 1	100	100	100	100	100	100
Level 2	77 64	76 63	105 88	64 53	83 69	63 53
Level 3	59 49	18 15	24 20	21 18	70 58	16 13

TABLE 2

INTRA-CONCEPT TASK COMPOSITE--KINDERGARTEN SAMPLE

A. Total number of subjects at each task level.

	Relations			Classes	Number	
	Seriation	Correspondence	Sorting	Class Inclusion	Conservation of Number	Unit
Level 1	41	42	15	43	34	49
Level 2	6	16	42	11	8	11
Level 3	13	2	3	6	18	0

B. Percent of subjects performing at or beyond each task level. Numerals in upper left corner of each cell represent the number of subjects reflected in the cell percentage.

	Relations			Classes	Number	
	Seriation	Correspondence	Sorting	Class Inclusion	Conservation of Number	Unit
Level 1	60 100	60 100	60 100	60 100	60 100	60 100
Level 2	19 32	18 30	45 75	17 28	26 43	11 18
Level 3	13 22	2 3	3 5	6 10	18 30	0 0

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TABLE 3

INTRA-CONCEPT TASK COMPOSITE--THIRD GRADE SAMPLE

A. Total number of subjects at each task level.

	Relations: Seriation	Relations: Correspondence	Classes: Sorting	Classes: Class Inclusion	Number: Conservation of Number	Number: Unit
Level 1	2	2	0	13	3	7
Level 2	12	42	39	32	5	37
Level 3	46	16	21	15	52	16

B. Percent of subjects performing at or beyond each task level. Numerals in upper left corner of each cell represent the number of subjects reflected in the cell percentage.

	Relations: Seriation	Relations: Correspondence	Classes: Sorting	Classes: Class Inclusion	Number: Conservation of Number	Number: Unit
Level 1	60 100	60 100	60 100	60 100	60 100	60 100
Level 2	58 97	58 96	60 100	47 78	57 95	53 88
Level 3	46 77	16 26	21 35	15 25	52 87	16 27

Tables 4, 5, and 6 represent cross classifications of each pair of concept tasks compared within subjects by grade and combined sample. Gamma scores (reflecting the strength of association between the two measures) and the significance levels are indicated.[5] Significant Gamma scores for the composite sample indicate a strong monotonic association between tasks within each concept (Relations, $\underline{G} = .677$, $\underline{p} = .001$; Classes, $\underline{G} = .668$, $\underline{p} = .001$; Number, $\underline{G} = .916$, $\underline{p} = .001$). Forty-four to 51 percent of the subjects achieved equivalent rankings on both measures for each concept (Relations, 44 percent; Classes, 43 percent; Number, 51 percent). Of the remaining subjects, over 79 percent scored higher on the first (basic understanding) than on the second (usage) task (Relations, 79 percent; Classes, 81 percent; Number, 100 percent).

When the subject sample is partitioned by grade levels, the strength of association is considerably deflated for the relational and classificatory tasks, while the consistency of performance within concept tasks remains high for all subjects at both grade levels.

Relations. An insignificant Gamma score of -.167 was indicated by the third grade sample.[6] The Gamma score for the kindergarten sample was .560 and significant at the .005 level. Equivalent level scores were recorded for 58 percent of the kindergarten subjects and 30 percent of the third grade sample. Of the remaining sample, 86 percent of the third grade subjects and 68 percent of the kindergarten subjects scored higher on seriation than they did on correspondence.

Classes. Third graders registered a significant Gamma value on class tasks while kindergarteners did not ($\underline{G} = .556$, $\underline{p} = .006$ versus $\underline{G} = .316$, $\underline{p} = .096$). Fifty-two percent of subjects in the third grade and only 33 percent of the subjects in the kindergarten sample scored at equivalent levels on both tasks. Of the remaining kindergarten subjects, 83 percent performed better on the sorting task than they did on the Class Inclusion task. Seventy-nine percent of the remaining third grade sample performed in a similar manner.

Number. Near equivalent and strong significant Gamma scores were recorded for both the third grade and kindergarten samples (.845, $\underline{p} = .001$; and .864, $\underline{p} = .003$, respectively). While 63 percent of the kindergarteners and 40 percent of the third graders scored at equivalent performance levels for both concept tasks, the balance of the subjects in each sample scored higher on conservation of number than on the unit task.

The tabulations in subtable D for each concept depict the placement of the subjects' responses into a stage assignment.

Inter-Concept Analysis

Conceptual stage designations for each grade level and the total sample are presented in Tables 7, 8, and 9. Subtable A reflects the number

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[5]Although the probability index does not consider tied rankings, it is reliable for the probability of Gamma and gains efficiency with large samples (see Goodman & Kruskal, 1954, 1959, 1963). Significance was determined at the .05 level.

[6]This is a function of the preponderance of subjects scoring at the third level on seriation and the second level on correspondence, thus leaving few untied pairs for Gamma comparisons.

TABLE 4

CROSS CLASSIFICATION OF CONCEPT TASKS--RELATIONS

A. Kindergarten Sample

Seriation Levels

Correspondence Levels	1	2	3	
1	33	4	5	42
2	6	2	8	16
3	2	0	0	2
	41	6	13	

$\underline{G} = .560$ $\underline{p} = .005$
Equivalent task levels = 58%

B. Third Grade Sample

Seriation Levels

Correspondence Levels	1	2	3	
1	0	1	1	2
2	1	7	34	42
3	1	4	11	16
	2	12	46	

$\underline{G} = -.167$ $\underline{p} = .274$
Equivalent task levels = 30%

C. Combined Sample

Seriation Levels

Correspondence Levels	1	2	3	
1	33	5	6	44
2	7	9	42	58
3	3	4	11	18
	43	18	59	

$\underline{G} = .677$ $\underline{p} = .001$
Equivalent task levels = 44%

D. Stage Assignments by Task Performance*

Seriation Level (X), Correspondence Level (Y), Overall Relational Stage (Z)

XYZ		XYZ		XYZ	
$\boxed{111}$	33/33	211	4/4	311	4/5
112		212	0/1	312	1/1
113		213		313	
121	6/7	221		321	
122	0	$\boxed{222}$	2/9	322	8/42
123		223		323	0
131	2/3	231		331	
132		232	0/4	332	
133		233	0	$\boxed{333}$	0/11

*Kindergarten sample/Total population

TABLE 5

CROSS CLASSIFICATION OF CONCEPT TASKS--CLASSES

A. Kindergarten Sample

Class Inclusion Levels	Sorting Levels 1	2	3	
1	11	32	0	43
2	3	7	1	11
3	1	3	2	6
	15	42	3	

G = .316 p = .096
Equivalent task levels = 33%

B. Third Grade Sample

Class Inclusion Levels	Sorting Levels 1	2	3	
1	0	11	2	13
2	0	22	10	32
3	0	6	9	15
	0	39	21	

G = .556 p = .006
Equivalent task levels = 51%

C. Combined Sample

Class Inclusion Levels	Sorting Levels 1	2	3	
1	11	43	2	56
2	3	29	11	43
3	1	9	11	21
	15	81	24	

G = .668 p = .001
Equivalent task levels = 42%

D. Stage Assignments by Task Performance*

Sorting Level (X), Class including Level (Y), Overall Class Stage (Z)

XYZ		XYZ		XYZ	
111	11/11	211	23/28	311	0
112		212	9/15	312	0/2
113		213		313	
121	2/2	221		321	
122	1/1	222	7/29	322	1/11
123		223		323	
131		231		331	
132	1/1	232	3/9	332	
133		233		333	2/11

*Kindergarten sample/Total population

TABLE 6

CROSS CLASSIFICATION OF CONCEPT TASKS--NUMBER

A. Kindergarten Sample

Conservation of Number Levels

Unit Levels	1	2	3	
1	$\boxed{34}$	5	10	49
2	0	$\boxed{3}$	8	11
3	0	0	$\boxed{0}$	0
	34	8	18	

$\underline{G}$ = .864 $\underline{p}$ = .001
Equivalent task levels = 61%

B. Third Grade Sample

Conservation of Number Levels

Unit Levels	1	2	3	
1	$\boxed{3}$	0	4	7
2	0	$\boxed{5}$	32	37
3	0	0	$\boxed{16}$	16
	3	5	52	

$\underline{G}$ = .845 $\underline{p}$ = .003
Equivalent task levels = 40%

C. Combined Sample

Conservation of Number Levels

Unit Levels	1	2	3	
1	$\boxed{37}$	5	15	57
2	0	$\boxed{8}$	39	47
3	0	0	$\boxed{16}$	16
	37	13	70	

$\underline{G}$ = .916 $\underline{p}$ = .001
Equivalent task levels = 51%

D. Stage Assignments by Task Performance*

Conservation of Number Level (X), Unit Level (Y), Overall Number Stage (Z)

XYZ		XYZ		XYZ	
$\boxed{111}$	34/37	211	5/5	311	8/10
112		212	0	312	2/5
113		213		313	
121		221		321	
122	0	$\boxed{222}$	3/8	322	8/39
123		223		323	
131		231		331	
132		232	0	332	
133		233		$\boxed{333}$	0/16

*Kindergarten sample/Total population

TABLE 7

CONCEPT STAGE COMPOSITE--TOTAL SAMPLE

A. Total number of subjects assigned to each concept stage.

	Relations	Classes	Number
Stage I	52	42	52
Stage II	57	67	52
Stage III	11	11	16

B. Percent of subjects assigned as in or passing each concept stage. Numerals in upper left corner of each cell represent the number of subjects reflected in the cell percentage.

	Relations	Classes	Number
Stage I	120 100	120 100	120 100
Stage II	68 57	78 65	68 57
Stage III	11 9	11 9	16 14

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TABLE 8

CONCEPT STAGE COMPOSITE--KINDERGARTEN SAMPLE

A. Total number of subjects assigned to each concept stage.

	Relations	Classes	Number
Stage I	49	36	47
Stage II	11	22	13
Stage III	0	2	0

B. Percent of subjects assigned as in or passing each concept stage. Numerals in upper left corner of each cell represent the number of subjects reflected in the cell percentage.

	Relations	Classes	Number
Stage I	60 100	60 100	60 100
Stage II	11 18	24 40	13 22
Stage III	0 0	2 3	0 0

TABLE 9

CONCEPT STAGE COMPOSITE--THIRD GRADE SAMPLE

A. Total number of subjects assigned to each concept stage.

	Relations	Classes	Number
Stage I	3	6	5
Stage II	46	45	39
Stage III	11	9	16

B. Percent of subjects assigned as in or passing each concept stage. Numerals in upper left corner of each cell represent the number of subjects reflected in the cell percentage.

	Relations	Classes	Number
Stage I	60 100	60 100	60 100
Stage II	57 95	54 90	55 92
Stage III	11 18	9 15	16 27

of subjects assessed at each stage, whereas subtable B indicates the percentage of subjects at or passing each concept stage.

The results for the stage designation tabulations indicate that an equal proportion of subjects attain or pass equivalent stage categories for all concepts. Differences are minor, when the number of subjects reflected in these proportions are considered. The most discrepant finding is at the kindergarten level where 40 percent (N = 24) of the sample were assigned in or beyond the second stage for classes, in comparison to 18 percent (N = 11) for relations and 22 percent (N = 13) for number. This difference was significant at the .002 level using the Cochran Q statistic (Q = 11.76).

The performance patterns for individual subjects on the three concepts are summarized in the cross classifications presented in Tables 10, 11, and 12.

The pair-wise concept comparisons for the composite sample resulted in the following findings:

1. A strong association between the concepts--

   Gamma = .816 for Relations x Classes

   Gamma = .875 for Relations x Number

   Gamma = .820 for Classes x Number

2. Roughly two-thirds of the subjects registered equivalent stages when comparing two concepts (subjects functioning at the same stage on both concepts).

   70 percent concordant rankings for Relations x Classes

   68 percent concordant rankings for Relations x Number

   65 percent concordant rankings for Classes x Number

3. The number of individuals showing discordant ranking on any two concepts is small. The discordant rankings do not reveal a clear sequence in the emergence of the three concepts. While the comparison of subjects demonstrating greater proficiency in one concept area over another does not result in a significant pattern, closer examination of individual cell comparisons reveals two phenomena.

   a. Although only 30 percent of the entire sample demonstrated discordant patterns between Stages I and II, more subjects demonstrated further developmental progress in the emergence of the class concept when compared to either relations or number concepts (with a binomial test, this result is significant at the .02 level). No discernible pattern of prior emergence was found for the developmental relationship between relations and number.

   b. Between Stage II and Stage III, the period before final conceptual consolidation, a sequential pattern among the emerging concepts is less evident.

When the subject sample is partitioned by grade level a slightly different developmental picture emerges. An analysis of subtables A and B for each pair-wise concept comparison reveals the following results:

TABLE 10

CROSS CLASSIFICATION OF THE CONCEPT AREAS--RELATIONS x CLASSES*

A. Kindergarten Sample

Class Stage	Relations Stage I	Relations Stage II	Relations Stage III	
I	35	2 1	0 0	36
II	23 14	8	0 0	22
III	0 0	3 2	0	2
	49	11	0	

$\underline{G} = .929$ $\underline{p} = .001$
72% concordant rankings

B. Third Grade Sample

Class Stage	Relations Stage I	Relations Stage II	Relations Stage III	
I	0	8 5	2 1	6
II	3 2	37	10 6	45
III	2 1	7 4	4	9
	3	46	11	

$\underline{G} = .259$ $\underline{p} = .156$
68% concordant rankings

C. Combined Sample

Class Stage	Relations Stage I	Relations Stage II	Relations Stage III	
I	35	5 6	1 1	42
II	13 16	45	5 6	67
III	1 1	5 6	4	11
	52	57	11	

$\underline{G} = .816$ $\underline{p} = .001$
70% concordant rankings

*Numerals in the upper left corner of the off-diagonal cells represent the percent of the subjects in that cell.

TABLE II

CROSS CLASSIFICATION OF THE CONCEPT AREAS--RELATIONS x NUMBER*

A. Kindergarten Sample

Number Stage	Relations Stage I	II	III	
I	41	10 6	0 0	47
II	13 8	5	0 0	13
III	0 0	0 0	0	0
	49	11	0	

G = .620 p = .017
77% concordant rankings

B. Third Grade Sample

Number Stage	Relations Stage I	II	III	
I	0	8 5	0 0	5
II	5 3	30	8 6	39
III	0 0	16 11	5	16
	3	46	11	

G = .515 p = .037
58% concordant rankings

C. Combined Sample

Number Stage	Relations Stage I	II	III	
I	41	9 11	0 0	52
II	9 11	35	4 6	52
III	0 0	8 11	5	16
	52	57	11	

G = .875 p = .001
68% concordant rankings

*Numerals in the upper left corner of the off-diagonal cells represent percent of the sample subjects in that cell.

TABLE 12

CLASS CLASSIFICATION OF THE CONCEPT AREAS--CLASSES x NUMBER*

A. Kindergarten Sample

Number Stage \ Class Stage	I	II	III	
I	31	25 15	2 1	47
II	8 5	7	2 1	13
III	0 0	0 0	0	0
	36	22	2	

$\underline{G}$ = .503 $\underline{p}$ = .032
63% concordant rankings

B. Third Grade Sample

Number Stage \ Class Stage	I	II	III	
I	3	3 2	0 0	5
II	5 3	32	7 4	39
III	0 0	18 11	5	16
	6	45	9	

$\underline{G}$ = .769 $\underline{p}$ = .001
67% concordant rankings

C. Combined Sample

Number Stage \ Class Stage	I	II	III	
I	34	14 17	1 1	52
II	7 8	39	4 5	52
III	0 0	9 11	5	16
	42	67	11	

$\underline{G}$ = .820 $\underline{p}$ = .001
65% concordant rankings

*Numerals in upper left corner of the off-diagonal cells represent percent of the sample subjects in that cell.

1. Except for one comparison significant (at the .05 level) Gamma values reflected the strong association between concepts.

Concepts	Kindergarten	Third Grade
Relations x Classes	G = .929*	.259
Relations x Number	.620*	.515*
Classes x Number	.503*	.769*

*Significant at the .05 level.

2. The proportion of concordant rankings between concepts by grade level is presented below:

Concepts	Kindergarten	Third Grade
Relations x Classes	72 percent	68 percent
Relations x Number	77 percent	58 percent
Classes x Number	63 percent	67 percent

While most concordant rankings in the kindergarten sample were recorded in the Stage I--I category, the large proportion of third graders recorded concordant rankings in the Stage II--II and Stage III--III categories (see Table 13).

3. The portion of subjects registering discordant rankings for the pair-wise concept comparison reveals differential grade level developmental patterns (see Tables 10, 11, 12, and 13).

   At the kindergarten level, with only 35 percent of sample registering discordant concept rankings, the performance patterns suggest that within the Stage I to Stage II period, classes emerge prior to both relations (significant at the .001 level) and number (significant at the .02 level). A discernible pattern of emergence between number and relational concepts is not evident. Performance indicating conceptual understanding beyond Stage II in the kindergarten sample is not present to suggest a pattern of Stage II--Stage III acquisition.

   At the third grade level, there is not a clear developmental sequence between discordant cases within the period between Stage I and Stage II for the three concepts. Within the period between Stage II and Stage III the discordant individuals did not reflect a significant sequential pattern in the relationships among class, relations, and number concepts.

The distribution of subjects by conceptual stage designations for relations, classes, and number is presented in Tables 14, 15, and 16. The observed as well as the expected frequency for the possible stage assignments are recorded by combined sample (Table 14) and grade level (kindergarten, Table 15; third grade, Table 16).[7]

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[7]The expected frequency was computed as the proportional probability for the event $X_iY_jZ_k$ multiplied by the number of subjects in the sample. The probability was calculated by multiplying the proportion of subjects at stage i for concept X by the proportion of subjects at stage j for concept Y by the proportion of subjects at stage k for concept Z.

TABLE 13

DISTRIBUTION OF CONCEPTUAL STAGE DESIGNATIONS

		Stage I	Between*	Stage II	Between*	Stage III
Relations x Classes	Kindergarten	35	15	8	2	0
	Third Grade	0	7	37	10	4
Relations x Number	Kindergarten	41	14	5	0	0
	Third Grade	0	8	30	17	5
Classes x Number	Kindergarten	31	20	7	1	0
	Third Grade	3	5	32	15	5

*Between-stage status indicates that the subject's performances on both concept assessments were discordant, hence demonstrating either Conceptual Stage I--II or II--III responses. Subjects demonstrating Conceptual Stage I--III responses are not included in this table.

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TABLE 14

DEVELOPMENTAL CONCEPT PATTERN--TOTAL SAMPLE

Concept Pattern XYZ	Observed	Expected	P(XYZ)*	Concept Pattern XYZ	Observed	Expected	P(XYZ)*	Concept Pattern XYZ	Observed	Expected	P(XYZ)*
$\boxed{111}$	31	7.9	.06	211	3	8.6	.07	311	0	1.6	.01
112	4	7.9	.06	212	3	8.6	.07	312	1	1.6	.01
113	0	2.4	.02	213	0	2.6	.02	313	0	.5	.004
121	10	12.6	.1	221	7	13.8	.11	321	0	2.6	.02
122	6	12.6	.1	$\boxed{222}$	29	13.8	.11	322	4	2.6	.02
123	0	3.9	.03	223	9	4.2	.04	323	2	.8	.007
131	0	2.0	.01	231	1	2.2	.02	331	0	.4	.003
132	1	2.0	.01	232	3	2.2	.02	332	1	.4	.003
133	0	.6	.005	233	2	.7	.006	$\boxed{333}$	3	.1	.001

53 percent concordant patterns

*Concepts XYZ: X = Relations
Y = Classes
Z = Number

TABLE 15

DEVELOPMENTAL CONCEPT PATTERN--KINDERGARTEN SAMPLE

Concept Pattern XYZ	Observed	Expected	P(XYZ)*	Concept Pattern XYZ	Observed	Expected	P(XYZ)*	Concept Pattern XYZ	Observed	Expected	-
111	31	23.0	.38	211	0	5.0	.09	311	0	0	-
112	4	6.3	.10	212	1	1.4	.02	312	0	0	-
113	0	0	-	213	0	0	-	313	0	0	-
121	10	14.1	.23	221	5	3.1	.05	321	0	0	-
122	4	3.9	.06	222	3	.9	.01	322	0	0	-
123	0	0	-	223	0	0	-	323	0	0	-
131	0	1.2	.02	231	1	.3	.004	331	0	0	-
132	0	.3	.005	232	1	.1	.001	332	0	0	-
133	0	0	-	233	0	0	-	333	0	0	-

57 percent concordant patterns

*Concepts XYZ: X = Relations
Y = Classes
Z = Number

TABLE 16

DEVELOPMENTAL CONCEPT PATTERN--THIRD GRADE SAMPLE

Concept Pattern XYZ	Observed	Expected	P(XYZ)*	Concept Pattern XYZ	Observed	Expected	P(XYZ)*	Concept Pattern XYZ	Observed	Expected	P(XYZ)*
$\boxed{111}$	0	0	.001	211	3	.4	.006	311	0	.1	.001
112	0	.2	.003	212	2	3.0	.05	312	1	.7	.01
113	0	.1	.001	213	0	1.2	.02	313	0	.3	.004
121	0	.2	.003	221	2	2.9	.05	321	0	.7	.01
122	2	1.5	.02	$\boxed{222}$	26	22.4	.32	322	4	5.4	.09
123	0	.6	.009	223	9	9.2	.15	323	2	2.2	.04
131	0	0	.001	231	0	.6	.01	331	0	.1	.002
132	1	.3	.004	232	2	4.5	.07	332	1	1.1	.02
133	0	.1	.001	233	2	1.8	.03	$\boxed{333}$	3	.4	.007

48 percent concordant patterns

*Concepts XYZ: X = Relations
Y = Classes
Z = Number

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Results of these tabulations indicate the following:

1. For the total sample, 63 subjects (53 percent) achieved equal stage status on the three concepts. These subjects were nearly equally distributed between kindergarten (34) and third grade (29) levels.
2. Except at the second conceptual stage for the kindergarten sample, the Cochran Q comparisons were insignificant. This indicates that equal proportions of subjects passed each concept at the same stage.
3. The observed frequency of concordant patterns for the three concepts is noticeably above expectation for both grade levels separately and as a total sample.
4. Although there were a large number of discordant patterns, no one pattern emerged. That is, no one sequential pattern was notably above expectation. While the data do not reveal a clearly discernible developmental sequence, Figures 3 and 4 illustrate the possible routes that might be taken by the emergent concepts in their ontogenesis.
5. A comparison of the concept patterns between the kindergarten (Table 15) and third grade (Table 16) reveals a greater degree of within group variability in the third grade group than in the kindergarten sample.[8] At the higher levels of concept acquisition, the third grade sample evidences greater diversity of ability than the kindergarten group.

ANALYSIS OF CONCEPTUAL PERFORMANCE ON BRAINERD'S NUMBER TASKS

The general performance patterns for Brainerd's concept tasks by grade level and for the composite sample are presented in Tables 17, 18, and 19. The figures in the first set of tabulations (A) reflect the number of subjects assessed at each stage of conceptual acquisition. The figures in the second set of tabulations (B) indicate the percentage and number (upper left corner of each cell) of subjects at or passing each concept stage.

The results of the stage designation tabulations indicate that the proportion of the subject sample attaining or passing a particular stage level are unequal across concept tasks. These differences are significant for both grades at the .05 level using the Cochran Q technique (at the second stage $Q = 35.5$ and 10.28 for kindergarten and third grade samples, respectively; at the third stage $Q = 40.62$ and 19.77 for kindergarten and third grade samples, respectively).

The performance patterns for individual subjects on the three concept tasks are summarized in the cross classifications presented in Tables 20, 21, and 22. An analysis by grade level for each pair-wise concept comparison reveals the following results:

1. The strength of association between concepts varies across grade levels. Gamma values are low with two comparisons registering significant patterns. The pattern of responses to Arithmetic and Cardinality at the kindergarten level and Arithmetic and Ordinality at the third grade level show significant monotone relationships.

[8] This reflects the floor effects evidenced in the kindergarten sample.

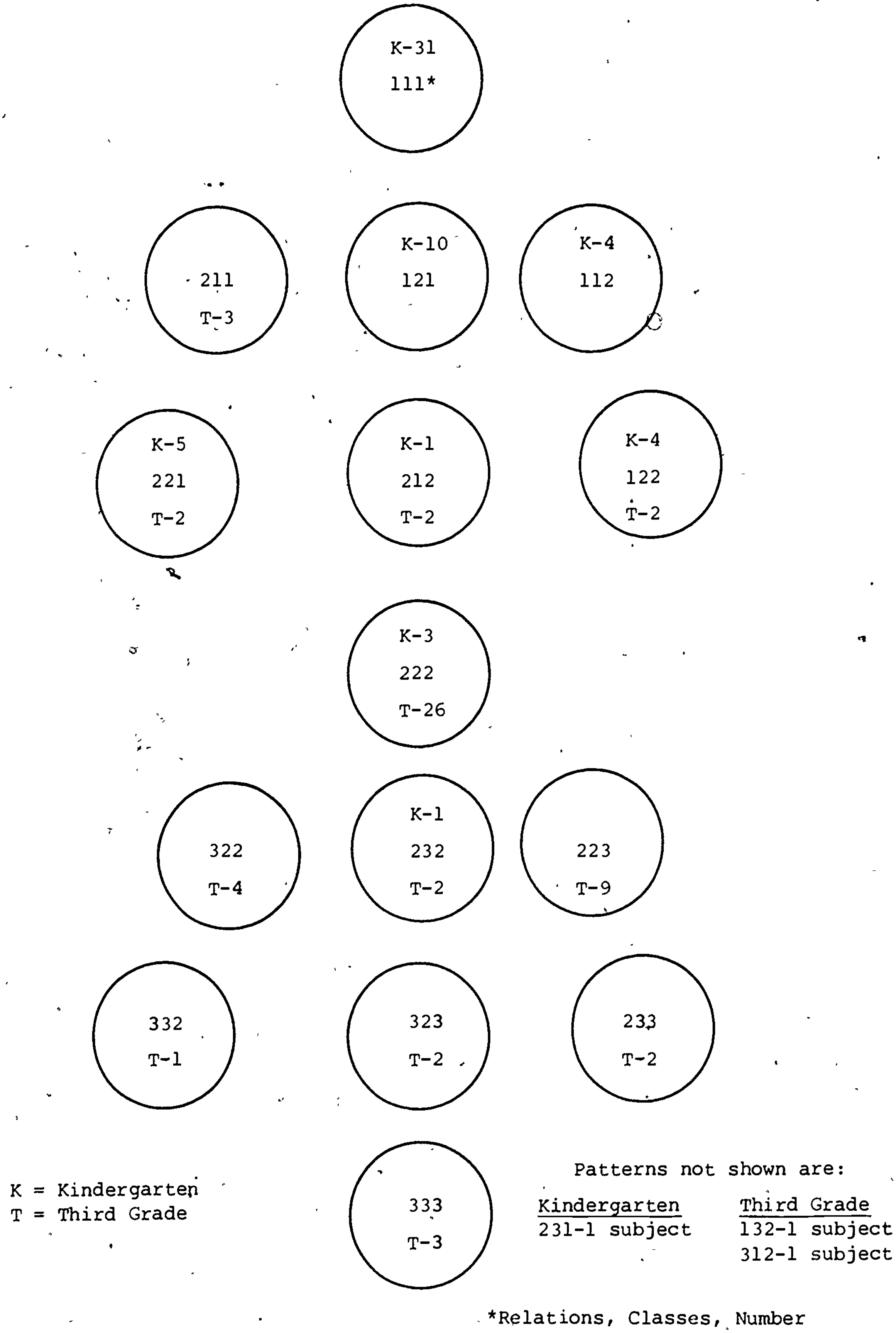

Figure 3. Concept stage patterns.

Kindergarten

	Alternative A	Alternative B	Alternative C	Alternative D
	** *** ****			
Stage I-II	121-Class-10		112-Number-4	
	122-Class-Number-4	221-Class-Relation-5	122-Class-4	212-Number-Relation-1
Stage II-III	Alternative A			
		231-Class-Relation-1		
	232-Class-1			

Third Grade

	Alternative A	Alternative B	Alternative C	Alternative D
				132-Class-Number-1
Stage I-II	211-Relation-3			
	221-Relation-Class-2	212-Relation-Number-2	122-Number-Class-2	
Stage II-III	Alternative A	Alternative B	Alternative C	Alternative D
		312-Relation-Number-1		
	322-Relation-4			232-Class-2
	332-Relation-Class-1	323-Relation-Number-2	233-Number-Class-2	

Alternative E

223-Number-9

Figure 4. Developmental patterns suggested by the present data.*

*Subjects responding with either full mastery (333), initial conception (III), or equivalent competence (222) are not included.
**Pattern as reflected in Tables 15 and 16.
***Concept(s) in advanced stage of development.
****Number of subjects demonstrating pattern.

TABLE 17

CONCEPT STAGE COMPOSITE--BRAINERD TASKS--TOTAL SAMPLE

A. Total number of subjects assigned to each concept stage.

	Ordinality	Cardinality	Arithmetic
Stage I	10	57	37
Stage II	34	47	19
Stage III	76	16	64

B. Percent of subjects assigned as in or passing each concept stage. Numerals in upper left corner of each cell represent the number of subjects reflected in the cell percentage.

	Ordinality	Cardinality	Arithmetic
Stage I	100	100	100
Stage II	110 92	63 53	83 69
Stage III	76 63	16 13	64 53

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TABLE 18

CONCEPT STAGE COMPOSITE--BRAINERD TASKS--KINDERGARTEN SAMPLE

A. Total number of subjects assigned to each concept stage.

	Ordinality	Cardinality	Arithmetic
Stage I	9	50	36
Stage II	21	10	15
Stage III	30	0	9

B. Percent of subjects assigned as in or passing each concept stage. Numerals in upper left corner of each cell represent the number of subjects reflected in the cell percentage.

	Ordinality	Cardinality	Arithmetic
Stage I	100	100	100
Stage II	51 85	10 17	24 40
Stage III	30 50	0 0	9 15

TABLE 19

CONCEPT STAGE COMPOSITE--BRAINERD TASKS--THIRD GRADE SAMPLE

A. Total number of subjects assigned to each concept stage.

	Ordinality	Cardinality	Arithmetic
Stage I	1	7	1
Stage II	13	37	4
Stage III	46	16	55

B. Percent of subjects assigned as in or passing each concept stage. Numerals in upper left corner of each cell represent the number of subjects reflected in the cell percentage.

	Ordinality	Cardinality	Arithmetic
Stage I	100	100	100
Stage II	59 98	53 88	59 98
Stage III	46 77	16 27	55 92

TABLE 20

CROSS CLASSIFICATION OF THE CONCEPT AREAS--ORDINALITY x CARDINALITY

A. Kindergarten

Cardinality Stage	Ordinality Stage I	II	III	
I	9	18	23	50
II	0	3	7	10
III	0	0	0	0
	9	21	30	

$\underline{G}$ = .515 $\underline{p}$ = .054
20% concordant rankings

B. Third Grade

Cardinality Stage	Ordinality Stage I	II	III	
I	1	0	6	7
II	0	10	27	37
III	0	3	13	16
	1	13	46	

$\underline{G}$ = .070 $\underline{p}$ = .404
40% concordant rankings

C. Total Sample

Cardinality Stage	Ordinality Stage I	II	III	
I	10	18	29	57
II	0	13	34	47
III	0	3	13	16
	10	34	76	

$\underline{G}$ = .488 $\underline{p}$ = .0007
30% concordant rankings

TABLE 21

CROSS CLASSIFICATION OF THE CONCEPT AREAS--ORDINALITY x ARITHMETIC

A. Kindergarten

Arithmetic Stage	Ordinality Stage I	II	III	
I	$\boxed{8}$	12	16	36
II	0	$\boxed{5}$	10	15
III	1	4	$\boxed{4}$	9
	9	21	30	

G = .240 p = .121
28% concordant rankings

B. Third Grade

Arithmetic Stage	Ordinality Stage I	II	III	
I	$\boxed{0}$	0	1	1
II	0	$\boxed{3}$	1	4
III	1	10	$\boxed{44}$	55
	1	13	46	

G = .650 p = .029
78% concordant rankings

C. Total Sample

Arithmetic Stage	Ordinality Stage I	II	III	
I	$\boxed{8}$	12	17	37
II	0	$\boxed{8}$	11	19
III	2	14	$\boxed{48}$	64
	10	34	76	

G = .472 p = .004
53% concordant rankings

TABLE 22

CROSS CLASSIFICATION OF THE CONCEPT AREAS--ARITHMETIC x CARDINALITY

A. Kindergarten

Cardinality Stage	Arithmetic Stage I	II	III	
I	$\boxed{33}$	10	7	50
II	3	$\boxed{5}$	2	10
III	0	0	$\boxed{0}$	0
	36	18	6	

G = .489 p = .030
63% concordant rankings

B. Third Grade

Cardinality Stage	Arithmetic Stage I	II	III	
I	$\boxed{1}$	0	6	7
II	0	$\boxed{3}$	34	37
III	0	1	$\boxed{15}$	16
	1	4	55	

G = .256 p = .267
31% concordant rankings

C. Total Sample

Cardinality Stage	Arithmetic Stage I	II	III	
I	$\boxed{34}$	10	13	57
II	3	$\boxed{8}$	36	47
III	0	1	$\boxed{15}$	16
	37	19	64	

G = .848 p = .0001
47% concordant rankings

Concepts	Kindergarten	Third Grade
Ordinality x Cardinality	G = .515	.070
Ordinality x Arithmetic	.240	.650*
Arithmetic x Cardinality	.489*	.256

*Significant at the .05 level.

2. The proportion of concordant rankings between concepts are low at both grade levels.

Concepts	Kindergarten	Third Grade
Ordinality x Cardinality	20 percent	40 percent
Ordinality x Arithmetic	28 percent	78 percent
Arithmetic x Cardinality	63 percent	31 percent

3. The portion of subjects at each grade level registering discordant rankings for the pair-wise comparisons reveals a consistent sequence in the emergence of the three concepts of ordinality, number, and cardinality. Ordinality is of significantly lesser difficulty than cardinality at the second and third stages for both kindergarteners and third graders.[9] Ordinality is significantly less difficult than arithmetic proficiency at both the second and third stages for the kindergarten sample and at the third stage for the third grade sample.[9] At both the kindergarten and third grade levels, arithmetic proficiency is of significantly less difficulty than cardinality at both the second and third stages.[9]

The distribution of subjects by conceptual stage designation for ordinality, cardinality, and number is presented for the total sample in Table 23 and for each grade in Tables 24 and 25. The developmental relationships existing among the three concepts, as suggested by the patterns evident in the subjects' task performances, are as follows:

1. For the total sample, 24 subjects (or 20 percent) achieved equal stage status on the three concepts. Eight subjects (or 13 percent) of the kindergarten sample and 16 subjects (or 26 percent) of the third grade sample registered concordant stage status across the three concepts.
2. Seventy-three percent of the kindergarten subjects responded in a number of discordant patterns which reflected the ordinal-arithmetic-cardinal conceptual sequence. Fifty-three percent of the third grade sample responded in similar patterns suggesting the identical sequence (see Figure 5). When disregarding patterns suggesting either no progress (Stage I) or full mastery (Stage III) of all three concept tasks, 85 percent of the kindergartners and 67 percent of the third graders reveal the ordinal-arithmetic-cardinal sequence (see Figure 6).

[9] By a binominal test, these results are significant at the .05 level.

TABLE 23

DEVELOPMENTAL CONCEPT PATTERNS--BRAINERD TASKS--TOTAL SAMPLE

Concept Pattern XYZ	Observed	Expected	P(XYZ)*	Concept Pattern XYZ	Observed	Expected	P(XYZ)*	Concept Pattern XYZ	Observed	Expected	P(XYZ)*
$\boxed{111}$	8	1.4	.01	211	10	5.0	.04	311	16	11.0	.09
112	0	.7	.006	212	5	2.5	.02	312	5	5.7	.05
113	2	2.5	.02	213	3	8.6	.07	313	8	19.2	.16
121	0	1.2	.01	221	2	4.1	.03	321	1	9.2	.08
122	0	.6	.005	$\boxed{222}$	4	2.1	.02	322	5	4.7	.04
123	0	2.1	.02	223	8	7.1	.06	323	27	15.9	.13
131	0	.4	.003	231	0	1.4	.01	331	0	3.1	.03
132	0	.2	.002	232	0	.7	.006	332	1	1.6	.01
133	0	.7	.006	233	3	2.4	.02	$\boxed{333}$	12	5.4	.04

20 percent concordant patterns

*Concepts XYZ: X = Ordinality
Y = Cardinality
Z = Arithmetic

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TABLE 24

DEVELOPMENTAL CONCEPT PATTERNS--BRAINERD TASKS--KINDERGARTEN SAMPLE

Concept Pattern XYZ	Observed	Expected	P(XYZ)*	Concept Pattern XYZ	Observed	Expected	P(XYZ)*	Concept Pattern XYZ	Observed	Expected	P(XYZ)*
111	8	4.5	.07	211	10	10.5	.17	311	15	15.0	.25
112	0	1.8	.03	212	5	4.4	.07	312	5	6.2	.10
113	1	1.1	.02	213	3	2.6	.04	313	3	3.7	.06
121	0	.9	.01	221	2	2.1	.03	321	1	3.0	.05
122	0	.4	.00	222	0	.9	.01	322	5	1.2	.02
123	0	.2	.00	223	1	.5	.01	323	1	.7	.01
131	0	0	--	231	0	0	--	331	0	0	--
132	0	0	--	232	0	0	--	332	0	0	--
133	0	0	--	233	0	0	--	333	0	0	--

13 percent concordant patterns

*Concepts XYZ: X = Ordinality
Y = Cardinality
Z = Arithmetic

TABLE 25

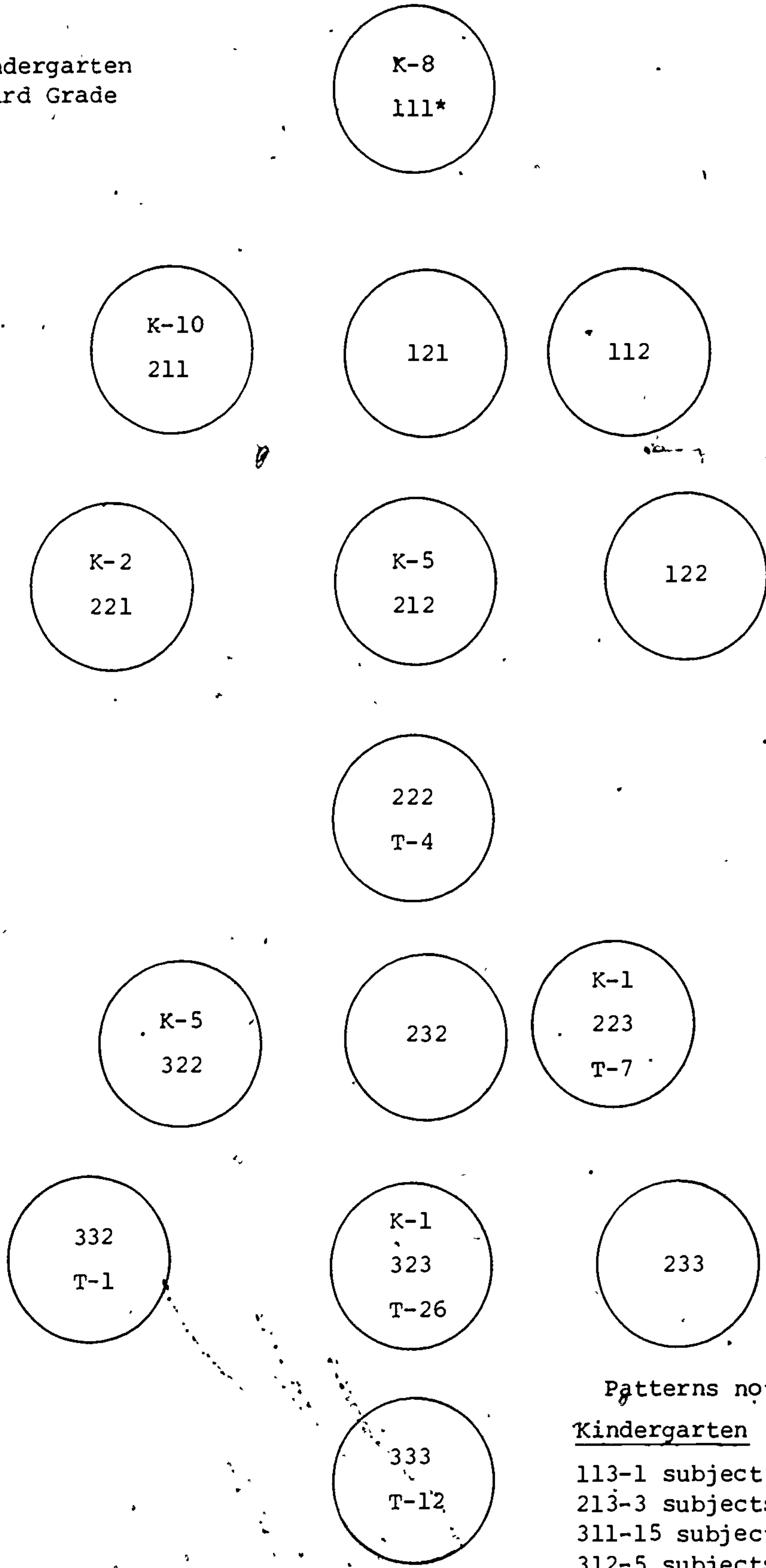
DEVELOPMENTAL CONCEPT PATTERNS--BRAINERD TASKS--
THIRD GRADE SAMPLE

Concept Pattern XYZ	Observed	Expected	P(XYZ)*	Concept Pattern XYZ	Observed	Expected	P(XYZ)*	Concept Pattern XYZ	Observed	Expected	P(XYZ)*
$\boxed{111}$	0	0	.00	211	0	0	.00	311	1	1.0	.00
112	0	0	.00	212	0	.1	.00	312	0	.4	.00
113	1	0	.00	213	0	1.0	.02	313	5	4.9	.08
121	0	0	.00	221	0	0	.00	321	0	.5	.00
122	0	0	.00	$\boxed{222}$	4	.5	.01	322	0	1.9	.03
123	0	1.0	.01	223	7	7.0	.12	323	26	25.9	.43
131	0	0	.00	231	0	.1	.00	331	0	.2	.00
132	0	0	.00	232	0	.2	.00	332	1	.8	.01
133	0	.2	.00	233	3	3.2	.05	$\boxed{333}$	12	11.2	.18

26 percent concordant patterns

*Concepts XYZ: X = Ordinality
Y = Cardinality
Z = Arithmetic

K = Kindergarten
T = Third Grade



Patterns not shown are:

Kindergarten	Third Grade
113-1 subject	113-1 subject
213-3 subjects	311-1 subject
311-15 subjects	313-5 subjects
312-5 subjects	
313-3 subjects	
321-1 subject	

*Ordinality, Cardinality, and Arithmetic

Figure 5. Concept stage patterns--Brainerd's measures.

Kindergarten

	Alternative A	Alternative B	Alternative C
	** *** ****		
Stage I-II	211-ordinality-10		
	212-ordinality-arithmetic-5	221-ordinality-cardinality-2	
Stage II-III	311-ordinality-15		113-arithmetic-1
	312-ordinality-arithmetic-5	321-ordinality-cardinality-1	213-arithmetic-ordinality-3
	313-ordinality-arithmetic-3		
	322-ordinality-5		223-arithmetic-1
	323-ordinality-arithmetic-1		

Third Grade

	Alternative A	Alternative B	Alternative C
Stage I-II			
Stage II-III	311-ordinality-1		113-arithmetic-1
	313-ordinality-arithmetic-5		223-arithmetic-7
	323-ordinality-arithmetic-26	332-ordinality-cardinality-1	233-arithmetic-cardinality-3

Figure 6. Developmental patterns suggested by the present data--Brainerd's measures.*

*Subjects responding with either full mastery (333), initial conception (111), or equivalent competence (222) are not included.

**Pattern as reflected in Tables 24 and 25.

***Concept(s) in advanced stage of development.

****Number of subjects demonstrating pattern.

ANALYSIS OF MAIN STUDY AND BRAINERD'S CONCEPTUAL TASK PERFORMANCE COMPARED

Pair-wise comparisons of performance on Brainerd's number tasks and performance on the composite concept tasks of the main assessment reveal the following results:

1. The cardinality task (which is identical to unit task in the main number battery) is strongly associated with all concepts in the main study. With only one exception, the cardinality task was found to be equally difficult as measures on the main study at all levels for both grade levels. The exception was at the kindergarten level where the class concept was significantly easier than the cardinality task at the second level.[10] These results are depicted in Table 26.

2. With one exception, ordinal task performance does not indicate a notable relationship to any main study concepts at either grade level. The exception is at the kindergarten level where ordinality is strongly associated with the concept of relations. The kindergarten sample found the ordinality task easier than any concept measure at both the second and third levels.[10] Similarly, the third-grade sample found the ordinal task to be of less difficulty than all main study concept areas at the third level.[10] These results are presented in Table 27.

3. The association between the arithmetic task and main study concepts is strongly established at the kindergarten level for number and relations. With one exception, the arithmetic task was found to be easier than the main study measures. The exception is at the second level for the class concept where no difference in performance is evident. For the third grade sample, arithmetic was of lesser difficulty than the main study measures at the third level while no difference in task performance at the second level is evident. Table 28 depicts these results.

An additional analysis was conducted to further specify the one conceptual pattern that appeared to be unique to each set of assessment instruments (measures either embodied in the main study or those designed by Brainerd). As presented earlier in the set of tables labelled Developmental Concept Patterns, there are 27 performance configurations for each set of measures. Corresponding to prevailing theories of natural number, these patterns represent four distinct and mutually exclusive conceptual acquisition sequences. Patterns within each sequence differ only in terms of their levels of acquisition and may be categorized as follows:

1. 8 patterns reflecting the primordial appearance of relational (ordinal) competence.
2. 8 patterns indicating the primacy of class (cardinal) notions.
3. 6 patterns reflecting the synthesis of relations (ordinality) and classes (cardinality).
4. 5 patterns may be considered as indicating number understanding emerging in advance of and not in co-development with class (cardinal) and relational (ordinal) competencies.

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[10] By a binominal test, these results are significant at the .05 level.

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TABLE 26

CROSS CLASSIFICATION OF BRAINERD'S CARDINALITY TASK TO MAIN TASK PERFORMANCES

Kindergarten

Class Stage	Cardinality Stage I	II	III	
I	33	3	0	36
II	15	7	0	22
III	1	1	0	2
	49	11	0	

$\underline{G}$ = .670 $\underline{p}$ = .02
67% concordant rankings

Relations Stage	Cardinality Stage I	II	III	
I	42	7	0	49
II	7	4	0	11
III	0	0	0	0
	49	11	0	

$\underline{G}$ = .658 $\underline{p}$ = .044
77% concordant rankings

Number Stage	Cardinality Stage I	II	III	
I	47	0	0	47
II	2	11	0	13
III	0	0	0	0
	49	11	0	

$\underline{G}$ = 1.00 $\underline{p}$ = .0001
97% concordant rankings

Third Grade

Class Stage	Cardinality Stage I	II	III	
I	3	3	0	6
II	4	30	11	45
III	0	4	5	9
	7	37	16	

$\underline{G}$ = .745 $\underline{p}$ = .006
63% concordant rankings

Relations Stage	Cardinality Stage I	II	III	
I	0	3	0	3
II	7	28	11	46
III	0	6	5	11
	7	37	16	

$\underline{G}$ = .505 $\underline{p}$ = .036
55% concordant rankings

Number Stage	Cardinality Stage I	II	III	
I	5	0	0	5
II	2	37	0	39
III	0	0	16	16
	7	37	16	

$\underline{G}$ = 1.000 $\underline{p}$ = .0001
97% concordant rankings

TABLE 27

CROSS CLASSIFICATION OF BRAINERD'S ORDINALITY TASK TO MAIN TASK PERFORMANCES

Kindergarten

Relations Stage × Ordinality Stage

Relations Stage	I	II	III	
I	8	16	25	49
II	1	5	5	11
III	0	0	0	0
	9	21	30	

$\underline{G}$ = .958 $\underline{p}$ = .0001
22% concordant rankings

Class Stage × Ordinality Stage

Class Stage	I	II	III	
I	6	12	18	36
II	3	8	11	22
III	0	1	1	2
	9	21	30	

$\underline{G}$ = .038 $\underline{p}$ = .43
25% concordant rankings

Number Stage × Ordinality Stage

Number Stage	I	II	III	
I	8	17	22	47
II	1	4	8	13
III	0	0	0	0
	9	21	30	

$\underline{G}$ = .292 $\underline{p}$ = .15
20% concordant rankings

Third Grade

Relations Stage × Ordinality Stage

Relations Stage	I	II	III	
I	0	2	1	3
II	1	9	36	46
III	0	2	9	11
	1	13	46	

$\underline{G}$ = .358 $\underline{p}$ ≐ .13
30% concordant rankings

Class Stage × Ordinality Stage

Class Stage	I	II	III	
I	1	2	3	6
II	0	9	36	45
III	0	2	7	9
	1	13	46	

$\underline{G}$ = .318 $\underline{p}$ = .129
28% concordant rankings

Number Stage × Ordinality Stage

Number Stage	I	II	III	
I	1	0	4	5
II	0	10	29	39
III	0	3	13	16
	1	13	46	

$\underline{G}$ = .141 $\underline{p}$ = .316
40% concordant rankings

TABLE 28

CROSS CLASSIFICATION OF BRAINERD'S ARITHMETIC TASK TO MAIN TASK PERFORMANCES

Kindergarten

Arithmetic Stage

Number Stage	I	II	III	
I	33	8	6	47
II	3	7	3	13
III	0	0	0	0
	36	15	9	

G = .616 p = .002
67% concordant rankings

Arithmetic Stage

Relations Stage	I	II	III	
I	33	10	6	49
II	3	5	3	11
III	0	0	0	0
	36	15	9	

G = .580 p = .009
63% concordant rankings

Arithmetic Stage

Class Stage	I	II	III	
I	23	8	5	36
II	13	6	3	22
III	0	1	1	2
	36	15	9	

G = .209 p = .173
50% concordant rankings

Third Grade

Arithmetic Stage

Number Stage	I	II	III	
I	1	0	4	5
II	0	3	36	39
III	0	1	15	16
	1	4	55	

G = .315 p = .225
32% concordant rankings

Arithmetic Stage

Relations Stage	I	II	III	
I	0	1	2	3
II	1	3	42	46
III	0	0	11	11
	1	4	55	

G = .830 p = .055
23% concordant rankings

Arithmetic Stage

Class Stage	I	II	III	
I	0	0	6	6
II	1	4	40	45
III	0	0	9	9
	1	4	55	

G = .200 p = .395
22% concordant rankings

Listed in Table 29 are the performance patterns for each battery of three concept measures by sequence category with the number of subjects observed to respond in such a pattern. This information is reduced and presented in Table 30.

It is evident in reviewing Tables 29 and 30 that when the subjects' conception of natural number was assessed by measures of the main study the predominant (59 percent of the total sample) acquisition pattern supports the class-relations synthesis theory. In contrast, when the same subjects were assessed using the measures developed by Brainerd, 64 percent responded with performance patterns reflecting the ordinal sequence in the acquisition of the natural number concept. To determine whether the number of subjects conforming to either the predictions of Piaget's model or the findings of Brainerd were due to differential task difficulties among assessment instruments within each study, a statistical procedure comparing differences between observed performance patterns and the expected frequencies of these response forms to the Chi Square statistic was adapted. The expected frequency of occurrence for each theoretical sequence was obtained by combining the conditional probability for each component pattern computed for the total sample in each study (see Tables 14 and 23). By using total sample computations, which eliminates both floor and ceiling effects, a global difficulty level was obtained. Distributions of observed and expected frequencies of theoretical sequences for both studies are presented in Table 31.

A Chi Square Goodness of Fit test, comparing the frequency distributions of observed and expected occurrences of theoretical sequences for the main study, revealed that actual apportionment of performance patterns differed significantly from what would be anticipated by difficulty levels for each assessment situation ($\chi^2 = 47.2$, df = 3, $p < .001$). Contributing to this result was the fact that the occurrence of the conjunctive sequence of relations and classes was nearly double what would be expected on the basis of assessment difficulty. It should also be noted that the number of subjects demonstrating the relations sequence was half the expected figure. This suggests that the large number of subjects demonstrating the co-emergent class-relations sequence is not an effect of assessment difficulty and may, in fact, reflect the true developmental nature of the natural number concept.

A Chi Square Goodness of Fit comparison between observed and expected frequency distributions of theoretical sequences for the supplementary study disclosed a significant difference ($\chi^2 = 10.9$, df = 3, $p = .012$). The major factor contributing to this result is that the conjunctive sequence of classes and relations occurs more frequently than would be expected. While a large number of subjects performed in an ordinal sequence, nearly the entire amount would have been expected given that difficulty levels among assessment tasks favored such a sequence. This indicates that patterns of conceptual performance in response to Brainerd's assessment tasks may be a function of task difficulty rather than an outcome related to the complexity of the natural number concept.

TABLE 29

PERFORMANCE PATTERNS PARTITIONED BY THEORETICAL SEQUENCE

Concepts* XYZ	Main Study Tasks Kinder-garten	Main Study Tasks Third Grade	Main Study Tasks Total Sample	Brainerd Tasks Kinder-garten	Brainerd Tasks Third Grade	Brainerd Tasks Total Sample
Relations-Ordinal Patterns						
211	0	3	3	10	0	10
311	0	0	0	15	1	16
212	1	2	3	5	0	5
312	0	1	1	5	0	5
313	0	0	0	3	5	8
321	0	0	0	1	0	1
322	0	4	4	5	0	5
323	0	2	2	1	26	27
Total	1	12	13	45	32	77
Class-Cardinal Patterns						
121	10	0	10	0	0	0
122	4	2	6	0	0	0
131	0	0	0	0	0	0
231	1	0	1	0	0	0
132	0	1	1	0	0	0
232	1	2	3	0	0	0
133	0	0	0	0	0	0
233	0	2	2	0	3	3
Total	16	7	23	0	3	3
Relations-Class-Ordinal-Cardinal Patterns						
111	31	0	31	8	0	8
221	5	2	7	2	0	2
222	3	26	29	0	4	4
331	0	0	0	0	0	0
332	0	1	1	0	1	1
333	0	3	3	0	12	12
Total	39	32	71	10	17	27
Number Patterns						
112	4	0	4	0	0	0
113	0	0	0	1	1	2
123	0	0	0	0	0	0
213	0	0	0	3	0	3
223	0	9	9	1	7	8
Total	4	9	13	5	8	13

*X = relations-ordinality
Y = classes-cardinality
Z = number-arithmetic

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TABLE 30

DISTRIBUTION OF THEORETICAL SEQUENCES

Total Sample	Relations		Classes		Relations-Classes		Number	
	No.	%	No.	%	No.	%	No.	%
Main Study	13	11	23	19	71	59	13	11
Supplementary Study	77	64	3	2.5	27	22.5	13	11
	Ordinal		Cardinal		Ordinal-Cardinal		Number	

Kindergarten	Relations		Classes		Relations-Classes		Number	
	No.	%	No.	%	No.	%	No.	%
Main Study	1	1.6	16	26.6	39	65	4	6.6
Supplementary Study	45	75	0	0	10	16.7	5	8.3
	Ordinal		Cardinal		Ordinal-Cardinal		Number	

Third Grade	Relations		Classes		Relations-Classes		Number	
	No.	%	No.	%	No.	%	No.	%
Main Study	12	20	7	11.7	32	53.3	9	15
Supplementary Study	32	53.3	3	5	17	28.3	8	13.3
	Ordinal		Cardinal		Ordinal-Cardinal		Number	

TABLE 31

DISTRIBUTION OF OBSERVED AND EXPECTED FREQUENCY OF THEORETICAL SEQUENCES

Main Study Theory	Relations	Classes	Relations- Classes	Number
Observed	13	23	71	13
Expected	26.9	34.9	36.4	21

Supplementary Study Theory	Ordinal	Cardinal	Ordinal- Cardinal	Arithmetic
Observed	77	3	27	13
Expected	73.2	7.6	17.7	21

VIII

DISCUSSION AND CONCLUSIONS

DISCUSSION

This study has attempted to examine the structural determinants of the natural number concept and the structural interdependence of the elements within it. Three philosophical views of the concept of natural number have been both formally and operationally defined. Piaget's theory contends that the number concept is in co-development with and a resultant of the coordinated synthesis of the concept of class and the concept of relations. Developmental concurrence of number, classes, and relations is cited as evidence to support his contentions. The ordinal position of Peano which is supported by the findings of Brainerd is based on the primordial presence of relations. Developmental asynchrony with relationality as the initial emergent competence provides evidence in support of this philosophy. The cardinal position of Russell would be supported if it were found that in human conceptualization the concept of classes developed in advance of the concept of relations. The hypothesized class-relations sequence in the emergence and maturity of the natural number concept is of basic interest in this discussion. The consistency of conceptual performance patterns which reflect interconcept developmental concurrence or sequence will demonstrate the appropriateness of the mathematical philosophies to account for the natural construction of the number concept in human logical reasoning.

Developmental concurrence and sequence are ambiguous expressions (Flavell, 1971). Piaget's theory is unclear about the degree to which or the sense in which he believes that the concepts of relations, classes, and number develop in synchrony. While his writings often convey the impression that these notions march in lockstep fashion into a child's logico-behavioral repertoire *(structure d' ensemble)* there is also frequent mention in his later work of *décalage* (or systematic age gaps) in cognitive acquisitions and maturity. Piaget is neither specific nor clear about what is meant by simultaneous development of classes, relations, and number. Flavell presumes that Piaget means "that development proceeds by very small increments: tiny advances in one area (via the usual mechanism of decentration with progressive equilibration, etc.) pave the way for similar small advances in another; these advances then redound to the developmental advantages of the first area, and so the spiral continues through ontogenesis [Flavell, 1963, p. 318]."

Assuming that the genesis of cognitive items is a gradual, fairly time consuming, extended process, does concurrence then mean that the concepts of relations, classes, and number begin to develop at the same point in ontogenetic time, or that they complete development at the same time, or both? Even if one concept develops prior to and completes its development before another, is that necessarily evidence of a sequence? What meaning should be given to the portion of time spent in co-development? Despite differences in the points of initial emergence and final consolidation,

periods of concurrent growth among cognitive elements may provide the information essential to understanding the nature and development of the natural number concept. The question of sequence and concurrence shifts from one based on the temporal relations between conceptual elements to one of the essential characteristics defining the nature of the interaction of those elements.

Identification of developmental concurrence becomes a task of demonstrating that the concepts of classes, relations, and number are not separate and independent elements in the child's cognitive behavioral repertoire. While cognitive elements evolve in more or less temporal proximity, concurrence is evident in the functional role they perform in the reciprocal or cyclic incremental facilitation of each other's development. Flavell expresses this relation as the "reciprocal, bidirectional (hence, 'asequential') effects and influences on one another's growth [1972, p. 282]." Each developmental increment in any one of the cognitive acquisitions must be shown to function as a mediator on the incremental progression in the other cognitive acquisitions. Cognitive items may come to modify each other in the course of their co-development by extending, broadening, and generalizing their range of application.

The findings of the main study reveal a strong association between the number-related concepts as reflected in their pair-wise comparisons. The high proportion of concordant rankings on concept areas reflects the bidirectional or reciprocal nature of their relationship. The discordant cases between classes and relations indicate that neither could be described as a <u>necessary</u> developmental prerequisite for the other.

It would be conjecture to conclude from the location of developmental status with respect to these concepts that there are functional linkages and complex interrelations among them. It may well be that these cognitive elements are unrelated or only very distantly related despite their apparent simultaneous appearance. The conceptual patterns identified in this study may be temporally coincident representing concurrent levels of acquisition and maturity on several parallel but independent developmental tracks. The strong values of association registered between concepts, the high proportion of concordant rankings, and the lack of a clear sequence suggest that there are possible underlying relations between the concept of relations and the concept of classes in the construction of the natural number concept. A direct, meaningful, and substantive relationship between the developmental interaction of the constituent items in the number concept must be demonstrated beyond a mere cross-sectional analysis. Further empirical assessment of the nature and development of the natural number concept in the child's cognitive structure may be achieved by means of long range longitudinal designs (see Wohlwill, 1973) and transfer of training analysis (see Beilin, 1971).

The developmental diagnosis of the number concept describes its ontogeny as an extended process of concordant and discordant patternings (see Tables 14, 15, and 16). While some children acquire the set of cognitive elements in one concurrent pattern, others acquire them in one of a number of other patterns. This does not necessarily reflect the absence of any functional developmental connection among the elements or imply that there is an invariant sequence in their acquisition. While 50 percent of the subject sample demonstrated a synchronous pattern in the development of number-related concepts, the balance of the subjects

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demonstrated a number of alternate and potentially ambiguous sequential patterns. The sequence notion stressed by both the ordinal and cardinal positions demands performance uniformity across children (regularity) and a consistent developmental lag between the primordial notion and other related concepts across developmental stages (invariance). The order of acquisition within each sequence, whether it be classes-relations-number or relations-classes-number, must be regular across children (universal) and absolutely invariant across the individual child's cognitive ontogeny. The relevant data consisting of triads of cognitive developmental acquisitions do not reveal a high degree of sequential regularity across same-age samples or an invariant pattern between age levels. If there were a sequential order to the acquisition of the number concept, as suggested by the philosophies of Russell and Peano, then conceptual responses of same-age subjects would not show the sequence reversals evident in the present data. Both the lack of evidence in support of regularity (within age levels) and the consistency between age levels suggest that the sequences demonstrated are neither universal nor invariant. Thus the present data do not support either the notions of Peano or the formulations of Russell.

Piaget's theory of the construction of the natural number concept stresses performance uniformity across children in the same developmental stage and low intra-individual variability across concepts. Piaget has amply demonstrated how complex the growth of logico-mathematical concepts can be. The dialectical constructive nature of cognitive development makes the assumption of conceptual equivalence nearly impossible to demonstrate empirically. Noting this, Flavell provides a more general description of a synchronous pattern, "One would not expect the ensemble of such 'same-level' items to show really extreme developmental asynchronies, e.g., one item beginning to emerge at age four and another not until age twelve. On the other hand, it does not follow at all that such items emerge in tight concurrence, that is, within the same week, month, or year [Flavell, 1971, p. 442]."

The data from this study reveal incidences of co-development of associated concepts in which a universal and invariant sequence is <u>not</u> apparent. This discovery suggests a meaningful, potentially bidirectional connection between cognitive items related to number which may well be considered to develop synchronously. Individual performance variability across concepts may be attributable to factors of measurement, to individual differences, and to the nature of cognitive development.

Concrete assessment situations developed to diagnose the developmental status of cognitive competencies may possess different sensitivities and give differential impressions about the emergence and maturity of related concepts. "The available evidence certainly leads one to believe that tight synchronisms are probably few and far between. The best evidence would, of course derive from studies where at least some attempt was made to equate the tests [Flavell, 1971, p. 441]."

A substantial effort was made in this study to develop measures which would diagnose accurately, and in strict operational ways, whether a child possessed understanding of the logical content domains of classification, relationality, and number in his cognitive repertoire. Although conceptual task item structure differed, the scoring scheme and stage designation procedures were based on conceptual and operational definitions and were

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consistently applied. (Tables 7, 8, and 9 indicate the degree to which this procedure produced equivalent test sensitivities.)

The differential rates of developmental maturity among the number concepts revealed in the present data may well have been expected. The unique experiential histories of the subjects involved in this study may have had an effect on the way the testing procedures activated newly emergent competencies. For some individuals, environmental experiences may be relevant to the formation of one concept while having no concomitant effect on other related concepts. Repeated contacts with certain clusters of environmental inputs may advance the developmental status of a certain concept while only minimally affecting others.

The basic functional nature of the emerging and changing cognitive elements is masked by instability, inflexibility, and uncertainty. The interconnected nature of their relations may not be readily perceptible. "At this point in development, the item--while now genuinely 'there, in the system' is conceived as being exceedingly fragile and difficult to elicit, highly vulnerable to blockage by innumerable 'performance' factors (memory and attentional problems, interfering perceptual and conceptual sets, and the like) [Flavell, 1970, p. 1033]." Strict conceptual concurrence may not be realized before the cognitive elements consolidate, stabilize, and generalize. As the conceptual elements mature, they become free of the limitations of performance factors. A set of cognitive competencies then emerges as a reliably elicitable cognitive apparatus, integrated into a functional totality and employed in appropriate situations to solve conceptual problems (see Flavell & Wohlwill, 1969).

The results of the main study may be compared to the previous findings of Brainerd (1972, 1973a, 1973b, 1973c, 1974) and Brainerd and Fraser (1975) and to the results of the supplementary analysis. The developmental patterns revealed in the main study do not support the conceptual sequence found by Brainerd. In task settings operationally derived from the same mathematical models presented earlier in this paper (and admittedly different from those employed in the main study), he finds relational concepts to precede the emergence of class concepts. Ordinal understanding precedes arithmetic competence which in turn precedes cardinal skills. The results of the supplementary analysis employing Brainerd's measures suggest a developmental picture that approximates those findings. The composite performance of individuals on the three concept tasks reveals a high degree of sequential regularity across same-age samples and an invariant pattern between age levels which is in accord with Brainerd's findings and reflective of the ordinal theory of natural number.

The discrepancy between the findings of the main study and those results discovered in the supplementary investigations may be understood by analyzing the differing experimental procedures employed in the arrival of those conclusions. First, it appears that Brainerd's conceptual tasks do not possess equal measurement sensitivities as evidenced by the significant Q values using the Cochran technique. The sequence found when employing Brainerd's tasks may be the result of differential task difficulties. Certain task features can account for the relative ease of conceptual performance among Brainerd's measures. Discrepant task performances are not surprising when the visual and practice factors of the ordinality task, the abstract, nontangible property of the cardinality measure, and the influence of individual learning history involved in the arithmetic test are considered. When the performances on the supplementary task

battery are compared to the conceptual responses documented in the main study, except for the cardinality task. Brainerd's measures are significantly less difficult than the analogous measures in the main study's concept battery. Although the tasks employed in the main study are of greater difficulty, they are more interrelated than Brainerd's tasks are to each other. Second, Brainerd's tasks do not provide multiple evidence for conceptual acquisition through a number of assessment situations for each concept area. Finally, Brainerd's measures may not have reflected the competencies which they were intended to assess.

Brainerd's cardination task is similar to the measure used by Piaget in the assessment of conservation of number. Piaget had criticized Russell for the use of such a behavioral analog in the assessment of one-to-one correspondence of equivalent classes because it introduces the notion of unit into its solution. This would make Brainerd's cardinality logically more difficult, for it demands a mastered concept of number (class x relations) for its solution. These assumptions are validated by the fact that cardinality is the only task that is highly related to the concepts examined in the main study. Furthermore, Brainerd's task of arithmetic assessment as a measure of natural number competency can also be questioned by applying Piaget's criticism of numeral use measures as markers for number concept acquisition. As mentioned earlier, the manipulation of numerals can be memorized without having attained an underlying conception of number (see p. 13 for explanation).

CONCLUSIONS

The findings of the main study suggest a developmental picture in which the concepts of classes and relations evolve co-jointly and are accompanied by their coordination and inclusion within the natural number concept. While it cannot be conclusively stated that both classes and relations are absolutely indispensable prerequisites to the emergence of the natural number concept, the present data do suggest that the developmental maturity of number is not achieved without the inclusion, or at least co-development, of both classes and relations. The strong association between the concept of number and the concepts of classes and relations coupled with the high proportion of concordant patterns and lack of a universal and invariant sequence suggest that there exists a conjunctive relation between the successive acquisitions of the three concepts. The conceptual construction of classes and relations seems to be integrated and incorporated within the accomplishments made in the construction of the natural number concept. These findings are concordant with the formulations of Piaget concerning the development of natural number understanding in children of middle-childhood age. The degree of regularity and consistency of these apparently concurrent cognitive events should, of course, be carefully investigated more extensively by both longitudinal assessment and experimental manipulations.

In contrast, conceptual performance patterns in response to Brainerd's number task array reveal an ordinal-arithmetic-cardinal sequence in the acquisition of the natural number concept. This sequence may only reflect the fact that the relational concept task used is of significantly lesser

difficulty than that for the classificatory concept counterpart. When analyzing Brainerd's task battery, it becomes apparent that the observed sequence of conceptual acquisitions may not be a function of difficulty as determined by the increasingly more complex cognitive operations necessary for the next concept's appearance; rather, it may simply be a function of increasing test item difficulty unrelated to conceptual competency. This was not the case for the assessment measures administered in the main study. Since tasks designed for the main study were equated for assessment sensitivity, conceptual patterns detected were not merely a result of task difficulty.

It becomes evident, when discussing the results of this two-phase study, that experimental evidence bearing on the accuracy of Piaget's model depends heavily on the judicious choice of tasks by the behavioral investigator. Relationships between abstract statements defining the grouping and assessment instruments must be made explicit by means of rules that translate formulae into experimental tasks. Principles, properties, and elements of logic may assist in the construction of a conceptual task array by specifying necessary task features that most accurately correspond to particular theoretical formulations. Designed this way, assessment instruments may more truthfully mirror differences in cognitive complexity among conceptual acquisitions. Patterns of concordance or sequence underlying cognitive competencies may then be detected relatively free of this one source of experimental error.

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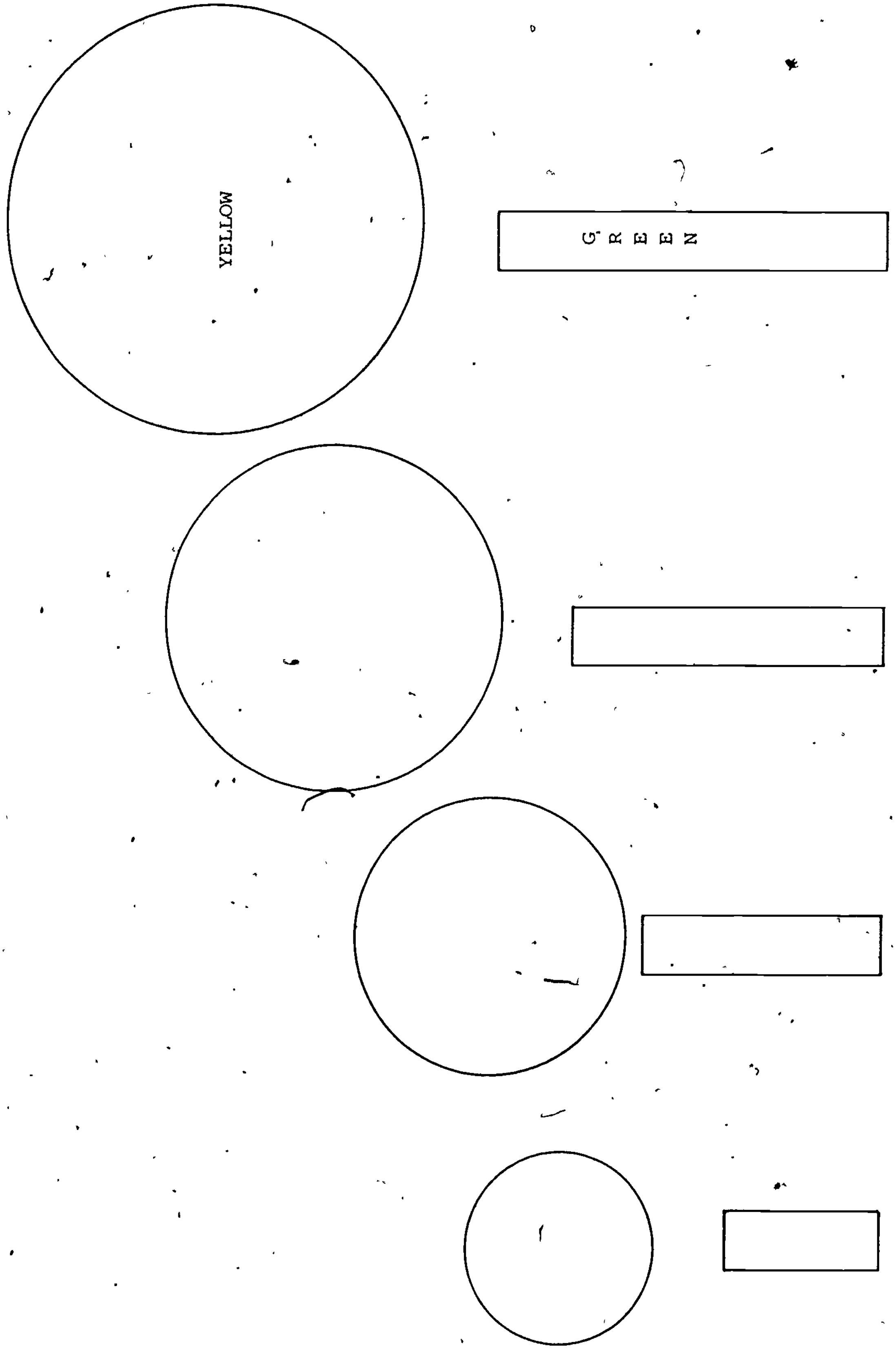
TASK MATERIALS AND PROTOCOLS

On the following pages are the task protocols and representations of task stimuli employed in this investigation:

Page

Note that the Conservation of Number stimuli are not represented in this section. Poker chips (ten red, ten blue) measuring one inch in diameter were used in the administration of this task.

SERIATION



Seriation

Warm-up Trial

E presents 4 sticks (3, 4, 5, 6) in scrambled fashion and not in a straight line.

HERE ARE SOME STICKS, I WANT YOU TO ARRANGE THEM IN ORDER.

If S hesitates E asks,

FIND THE SHORTEST STICK AND PLACE IT HERE (to the S's right) pause AND FIND THE LONGEST STICK AND PLACE IT HERE (to S's left). NOW PUT ALL THE OTHER STICKS IN ORDER BETWEEN THE LONGEST AND THE SHORTEST.

E completes task if necessary and asks,

WHY DID YOU (WE) ORDER THE STICKS THAT WAY?

E tries to get S to use the increasing length of the sticks as the criterion for the arrangement.

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Seriation

Ordering

E presents sticks (1, 3, 4, 6, 7, 8, 10) in a scrambled fashion and not in a straight line.

HERE WE HAVE SOME STICKS. I WANT YOU TO ARRANGE THESE STICKS IN ORDER. REMEMBER ALL THE STICKS HAVE TO BE IN ORDER.

If the child hesitates E asks,

FIND THE SHORTEST STICK AND PLACE IT HERE (to the S's right) pause AND FIND THE LONGEST STICK AND PLACE IT HERE (to the S's left). NOW PLACE ALL THE OTHER STICKS IN ORDER BETWEEN THE LONGEST AND THE SHORTEST.*

E helps S to finish the task if necessary.*

ARE YOU FINISHED? ARE THEY JUST RIGHT?

E arranges sticks in correct order before proceeding to next task.

*An incorrect ordering score is recorded for those sticks for which help was required.

Seriation

Prediction and Placement

E adjusts the array so that there is an inch space between the sticks and a two inch space between the sticks where a stick is to be added. Pause.

E places the remaining sticks (2, 5, 9) an inch apart between the original array and the S in order and in the same ascending direction as the original array.

Prediction:

HERE ARE THREE MORE STICKS THAT GO WITH THE OTHER STICKS. CAN YOU SHOW ME WITHOUT TOUCHING THE STICKS WHERE THIS STICK (pointing to 2) WOULD GO INTO THE ORDER OF STICKS? E repeats this procedure with stick 5, then 9.

Placement: (If E helps, score that stick as incorrect.)

NOW PUT THESE STICKS INTO THE ORDER WITH THE OTHER STICKS. PUT THEM IN WHERE THEY BELONG.

If S fails to understand the task, E places one of the sticks for him.

After S has finished E asks,

HAVE YOU PLACED THE STICKS THE WAY YOU WANT THEM? CHECK AND MAKE SURE.

Serial Correspondence

E places the circles between the S and the order array of sticks in a mixed fashion.

HERE ARE SOME CIRCLES, I WANT YOU TO ARRANGE THESE CIRCLES IN ORDER. REMEMBER ALL OF THE CIRCLES HAVE TO BE IN ORDER. PUT THEM IN THE SAME WAY AS THE STICKS.

If the child hesitates E asks,

FIND THE SMALLEST CIRCLE AND PLACE IT HERE (to the side of the shortest stick) pause AND FIND THE LARGEST CIRCLE AND PLACE IT HERE (to the side of the largest stick). NOW PLACE ALL THE OTHER CIRCLES IN ORDER BETWEEN THE LARGEST AND THE SMALLEST CIRCLE.*

The circles should be very close together but not touching.

NOW LET'S MATCH EACH STICK TO THE CIRCLE IT GOES WITH. Pause to see if S can make the correct correspondence order.

If he hesitates E asks,

PUT THE LONGEST STICK WITH THE LARGEST CIRCLE. Pause. PUT THE SHORTEST STICK WITH THE SMALLEST CIRCLE. NOW PUT EACH OF THESE STICKS WITH THE CIRCLE THAT IT GOES WITH. THEY MUST BE THE RIGHT SIZE FOR EACH OTHER.**

If S responds incorrectly E replaces the sticks and circles to correspondence and explains,

EACH STICK GOES WITH A CIRCLE. THEY ARE THE RIGHT SIZE FOR EACH OTHER.

E extends the stick array so there is an extra two sticks at each end of the circle array. Place sticks so the longest sticks are roughly 4 inches above the largest circles. Order sticks over spaces between circles. E points to stick 5 and asks,

POINT TO THE CIRCLE THAT GOES WITH THIS STICK. THEY HAVE TO BE THE RIGHT SIZE FOR EACH OTHER.

E repeats this procedure for sticks 2 and 7.

*An incorrect ordering score is recorded for those circles for which help was required.

**An incorrect correspondence score is recorded for those sticks for which help was required.

Serial Correspondence (Cont.)

S may not move the sticks or circles.

E returns the sticks and circles to correspondence and once again establishes the relationship between the circles and sticks.

E compresses sticks (4 inches above largest circle) so all the sticks are one inch apart and clustered between circles 4 and 8. E points to stick 6.

POINT TO THE CIRCLE THAT GOES WITH THIS STICK. THEY HAVE TO BE THE RIGHT SIZE FOR EACH OTHER.

E repeats this procedure for stick 8, then 3.

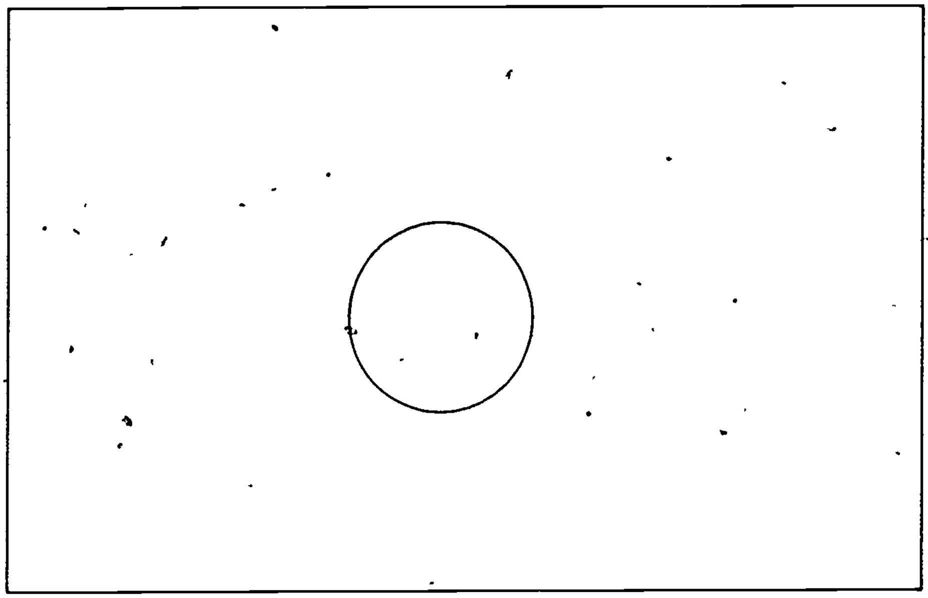
E returns the sticks and circles to correspondence and once again establishes the relationship between the circles and sticks.

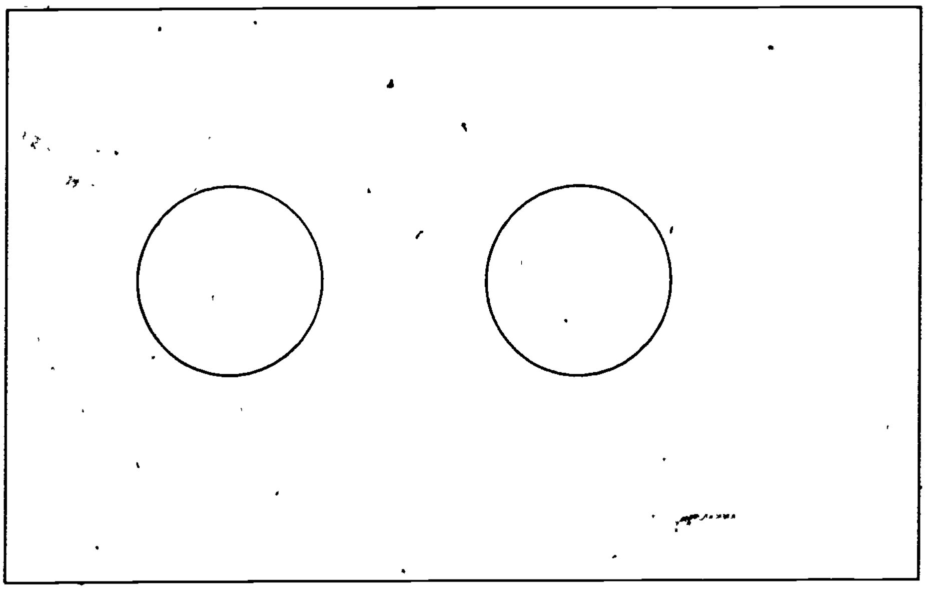
E scrambles the sticks. E points to stick 4 and asks,

WHAT CIRCLE GOES WITH THIS STICK? YOU CAN DO ANYTHING YOU WANT WITH THE STICKS. (Don't tell to construct the order even though it is permissible.) REMEMBER, THEY HAVE TO BE THE RIGHT SIZE FOR EACH OTHER.

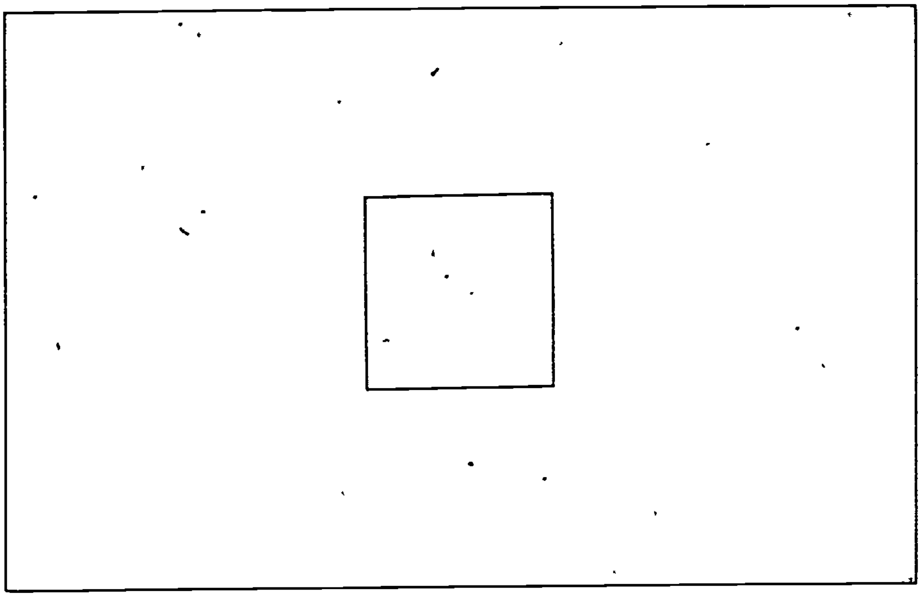
E repeats this procedure for stick 9, then 5. If S moves the sticks during a trial, E scrambles the sticks again.

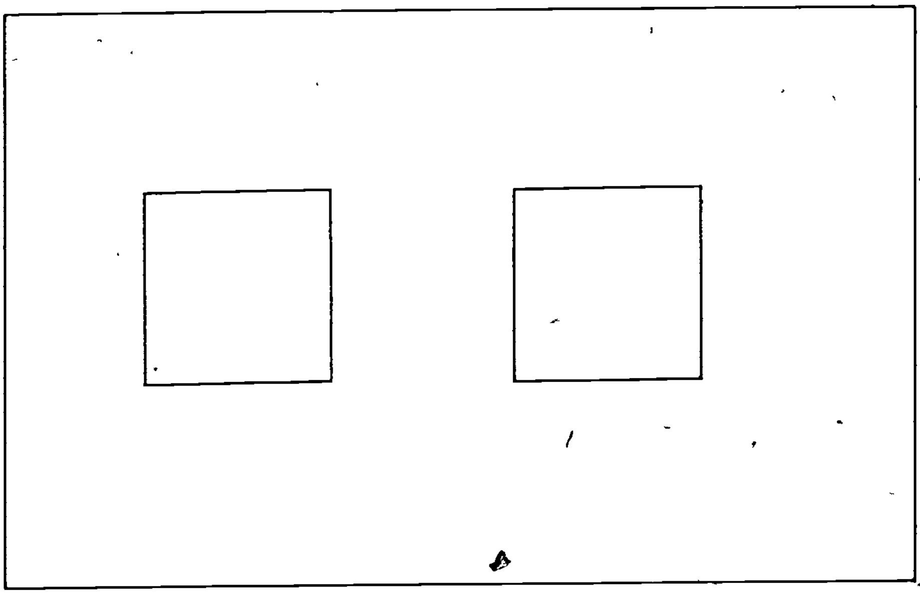
SORTING





* 3 Blue, 3 Red
** 3 Blue, 2 Red





* 3 Blue, 3 Red
** 2 Blue, 3 Red

Dichotomies

DIVIDE ALL THESE DRAWINGS INTO TWO BUNCHES. PUT ONE KIND HERE AND ONE KIND HERE. For 2nd and 3rd Dichotomies, E adds, BUT DO IT IN A DIFFERENT WAY THAN BEFORE.

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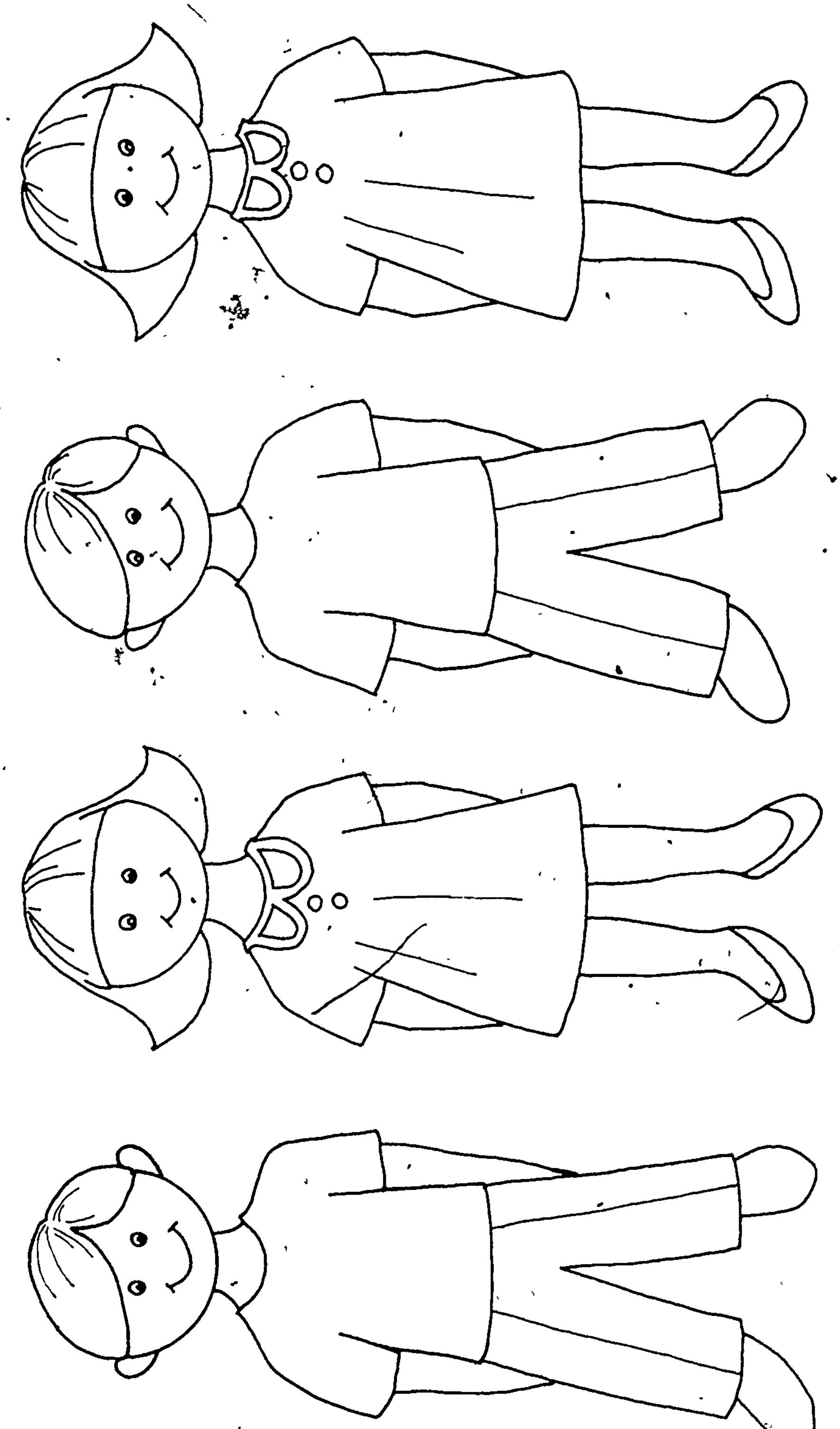
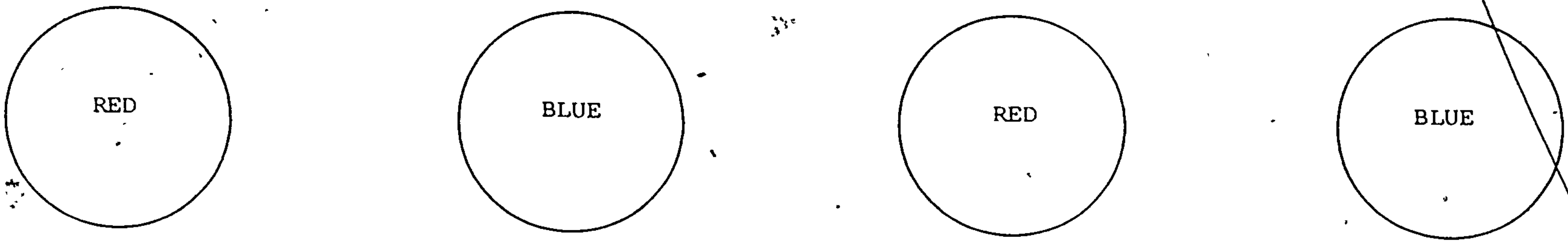
CLASS INCLUSION A



CLASS INCLUSION A



Class Inclusion A

Order of Presentation: 1. children--circles
2. circles--children

<u>Children Stimuli</u> (2 Boys, 2 Girls)

LOOK AT ALL THE <u>CHILDREN</u>.

ARE THERE MORE CHILDREN OR MORE BOYS?

more children,* more boys, other

ARE THERE FEWER BOYS OR FEWER CHILDREN?

fewer boys,* fewer children, other

COUNT THE CHILDREN.**

NOW, COUNT THE BOYS.**

NOW, COUNT THE GIRLS.**

LOOK AT ALL THE CHILDREN.

ARE THERE MORE CHILDREN OR MORE BOYS?

more children,* more boys, other

ARE THERE FEWER BOYS OR FEWER CHILDREN?

fewer boys,* fewer children, other

<u>Circular Stimuli</u> (2 Blue Circles, 2 Red Circles)

LOOK AT ALL THE CIRCLES.

ARE THERE MORE CIRCLES OR MORE BLUE CIRCLES?

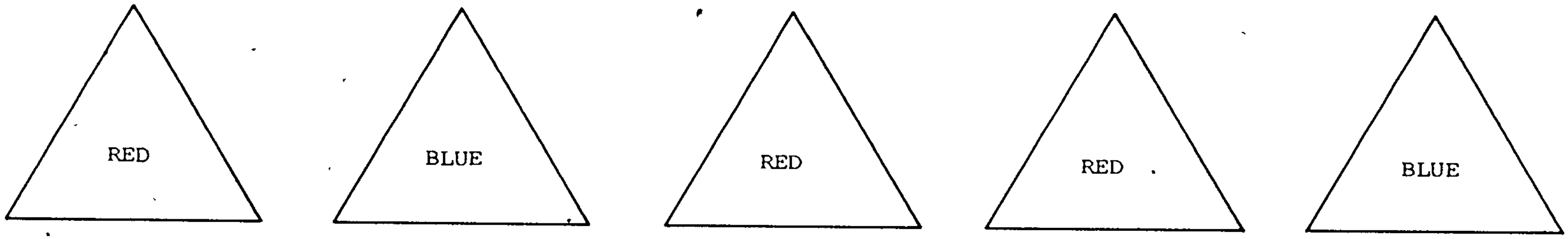
more circles,* more blues, other

ARE THERE FEWER BLUE CIRCLES OR FEWER CIRCLES?

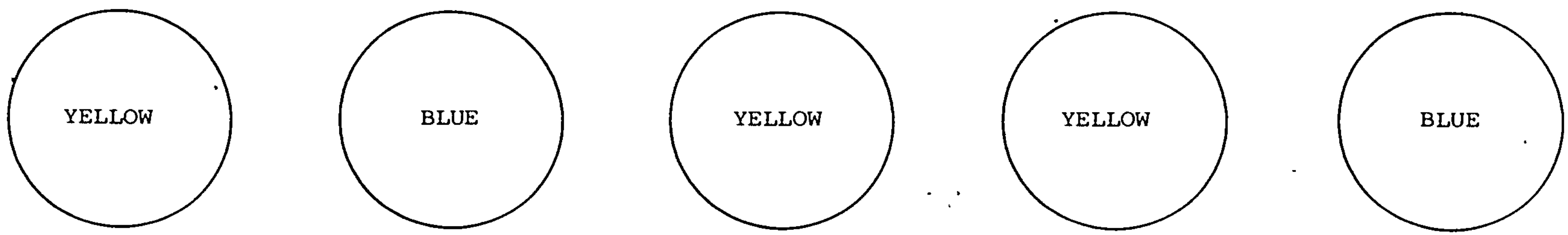
fewer blues,* fewer circles, other

*Correct Response
**<u>E</u> helps <u>S</u> to count stimuli if necessary.

CLASS INCLUSION B



CLASS INCLUSION B



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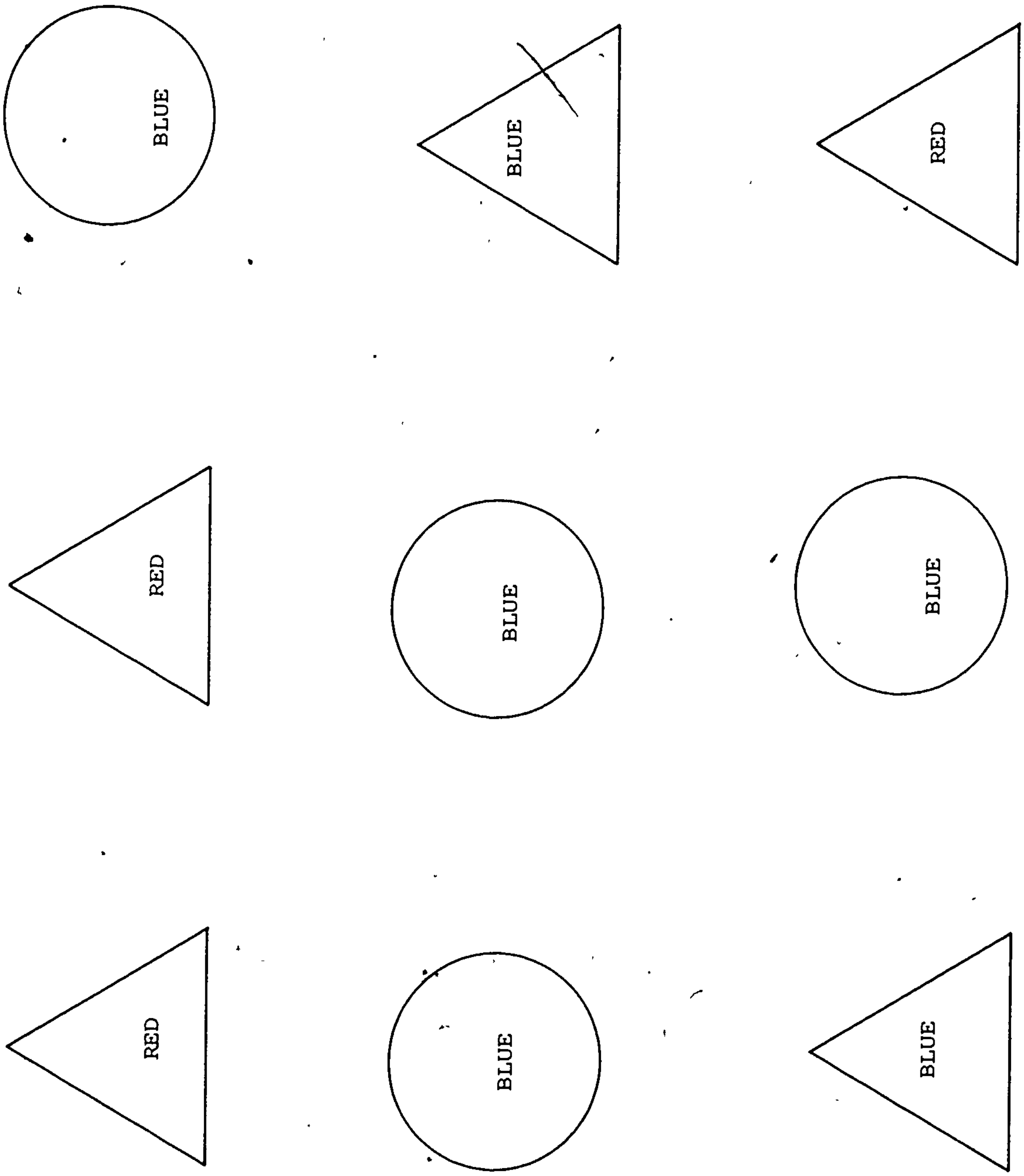
CLASS INCLUSION B



Class Inclusion B

1. Materials: 3 red and 2 blue triangles

   ARE THERE MORE TRIANGLES OR MORE RED FIGURES?

   more triangles,* more red figures, other

2. Materials: 3 yellow and 2 blue circles

   ARE THERE MORE BLUE FIGURES OR MORE CIRCLES?

   more circles,* more blue figures, other

3. Materials: 3 red and 2 blue triangles, 4 blue circles

   a. ARE THERE MORE TRIANGLES OR MORE RED FIGURES?

      more triangles,* more red figures, other

   b. ARE THERE MORE BLUE FIGURES OR MORE CIRCLES?

      more blue figures,* more circles, other

   c. ARE THERE MORE BLUE FIGURES OR MORE TRIANGLES?

      more blue figures,* more triangles, other

*Correct response

ERIC

Conservation of Number

Materials:

20 plastic chips

Procedure: The experimenter and subject construct two parallel rows of evenly spaced chips in the center of the table. There is a precise perceptual correspondence between the elements of the two rows.

1. Prediction: Leaving the rows exactly as they are, the experimenter asks the following questions:

   a. IF I WERE TO PUSH THE CHIPS IN THIS ROW (pointing to the row nearest the experimenter) VERY CLOSE TOGETHER, WOULD THE TWO ROWS STILL HAVE THE SAME NUMBER OF CHIPS?

   b. IF I WERE TO PUSH THE CHIPS IN THIS ROW (indicating the same row) VERY CLOSE TOGETHER, WOULD ONE OF THE ROWS HAVE MORE CHIPS?

   c. IF I WERE TO PUSH THE CHIPS IN THIS ROW (indicating the same row) VERY CLOSE TOGETHER, WOULD ONE OF THE ROWS HAVE FEWER CHIPS?

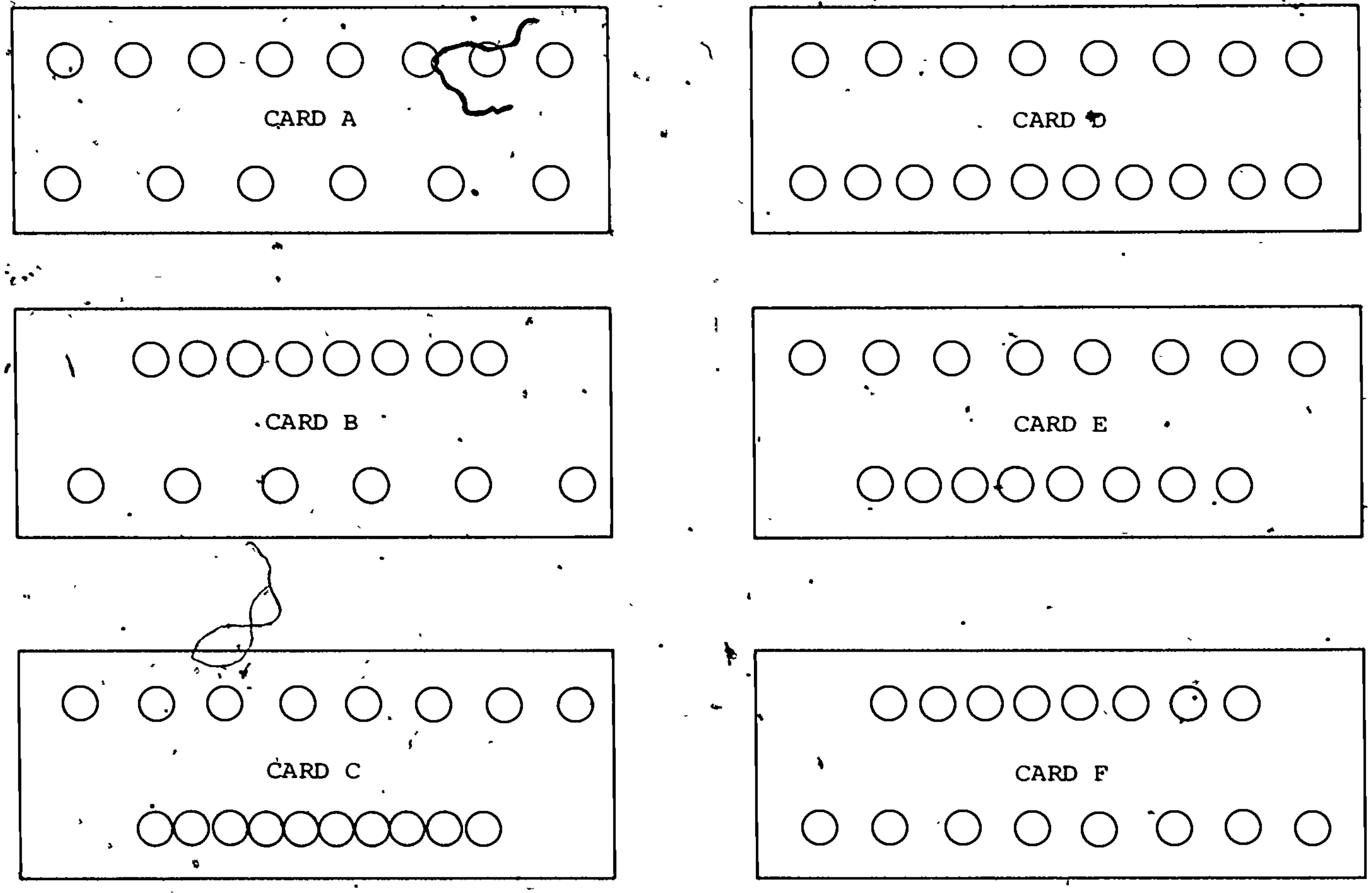
2. Deformation: The experimenter pushes the chips in the nearest row together until they touch. The row nearest the subject is now roughly three times as long as the other row. The experimenter asks the following (randomly ordered) questions:

   a. DO THESE TWO ROWS HAVE THE SAME NUMBER OF CHIPS?

   b. DOES ONE OF THE ROWS HAVE MORE CHIPS NOW?

   c. DOES ONE OF THE ROWS HAVE FEWER CHIPS NOW?

UNIT (Cardinality)



TOP ROW RED, BOTTOM ROW GREEN

Unit (Cardinality)

NOW WE ARE GOING TO PLAY A GAME. I AM GOING TO SHOW YOU SOME CARDS WITH TWO ROWS OF DOTS ON THEM. I WANT YOU TO FIGURE OUT WHETHER THERE ARE THE SAME NUMBER OF RED DOTS AS GREEN DOTS, OR IF ONE OF THE TWO ROWS HAS MORE DOTS.

ONE SPECIAL RULE YOU MUST FOLLOW--YOU CANNOT COUNT THE DOTS--YOU HAVE TO FIGURE OUT THE ANSWER SOME OTHER WAY.

One at a time E presents a card on the table in front of S with the blue dots closest to S. S may not touch the card. E asks the questions below for each card. Allow as much time as needed.

Card A (Red = 8 dots, 11" long; Green = 6 dots, 11" long.)

ARE THERE THE SAME NUMBER OF RED DOTS AS GREEN DOTS ON THIS CARD?

No* Yes

DOES ONE OF THE ROWS HAVE MORE DOTS?

Yes* No

If yes, WHICH ONE? Red* Green
If no,

HERE IS A NEW CARD.

Card B (Red = 8 dots, 8" long; Green = 6 dots, 11" long.)

ARE THERE THE SAME NUMBER OF RED DOTS AS GREEN DOTS ON THIS CARD?

No* Yes

DOES ONE OF THE ROWS HAVE FEWER DOTS?

Yes* No

If yes, WHICH ONE? Green* Red
If no,

HERE IS A NEW CARD,

Card C (Red = 8 dots, 11" long; Green = 10 dots, 8" long.)

ARE THERE THE SAME NUMBER OF RED DOTS AS GREEN DOTS ON THIS CARD?

No* Yes

DOES ONE OF THE ROWS HAVE MORE DOTS?

Yes* No

*Correct response

Unit (Cardinality) (Cont.)

If yes, WHICH ONE? Green* Red
If no,

HERE IS A NEW CARD.

Card D (Red = 8 dots, 11" long; Green = 10 dots, 11" long.)

ARE THERE THE SAME NUMBER OF RED DOTS AS GREEN DOTS ON THIS CARD?

No* Yes

DOES ONE OF THE ROWS HAVE FEWER DOTS?

Yes* No

If yes, WHICH ONE? Red* Green
If no,

HERE IS A NEW CARD.

Card E (Red = 8 dots, 11" long; Green = 8 dots, 8" long.)

ARE THERE THE SAME NUMBER OF RED DOTS AS GREEN DOTS ON THIS CARD?

Yes* No

DOES ONE OF THE ROWS HAVE MORE DOTS?

No* Yes

HERE IS A NEW CARD.

Card F (Red = 8 dots, 8" long; Green = 8 dots, 11" long.)

ARE THERE THE SAME NUMBER OF RED DOTS AS GREEN DOTS ON THIS CARD?

Yes* No

DOES ONE OF THE ROWS HAVE FEWER DOTS?

No* Yes

*Correct response

ERIC

Ordinality--Length

Materials:

27-cm blue stick
28-cm blue stick
28-cm white stick

Instructions:

The E places the board, having a 27-cm blue stick and 28-cm blue stick glued down approximately one arm's length apart, 8-10 inches from the S in the middle of the table. The sticks are positioned such that the midpoint of each stick is in direct relation to the other stick. Taking the 28-cm white stick and placing it in the middle of the board between the two blue sticks, the E says:

HERE ARE SOME STICKS WE WILL BE WORKING WITH.

The E then places the 28-cm white stick next to the 28-cm blue stick, making the ends nearest the S even with one another, and so the S can observe the sticks to be of equal length. The S is required to verbalize this latter fact.

ARE THESE TWO STICKS THE SAME LENGTH?

Yes ___ No ___ I Don't Know ___ No Response ___

Next, the E places the 28-cm white stick next to the 27-cm blue stick, again making the ends nearest the S even with one another, and so the S can observe that the white stick is the longer of the two. The S is required to verbalize this latter fact.

IS ONE OF THE STICKS LONGER?

Yes ___ No ___ I Don't Know ___ No Response ___

(If "Yes," then) WHICH ONE?

White ___ Blue ___ I Don't Know ___ No Response ___

Ordinality--Length (Cont.)

Finally, the E removes the white stick from the table, and asks the following questions:

(a) ARE THESE TWO STICKS THE SAME LENGTH?

Yes ___ No ___ I Don't Know ___ No Response ___

(b) IS ONE OF THE STICKS LONGER?

Yes ___ No ___ I Don't Know ___ No Response ___

(If "Yes," then) WHICH ONE?

28-cm ___ 27-cm ___ I Don't Know ___ No Response ___

(c) IS ONE OF THE STICKS SHORTER?

Yes ___ No ___ I Don't Know ___ No Response ___

(If "Yes," then) WHICH ONE?

27-cm ___ 28-cm ___ I Don't Know ___ No Response ___

Ordinality--Weight

Materials:

One red and one gray clay ball of equal weight
One gray clay ball of a lighter weight

Instructions:

The E places the three clay balls in the middle of the table 8-10 inches from the S, and says:

HERE ARE SOME CLAY BALLS WE WILL BE WORKING WITH.

The E then hands the S one red and one gray clay ball of equal weight. The S is required to verbalize this latter fact.

DO THESE TWO CLAY BALLS WEIGH THE SAME?

Yes ___ No ___ I Don't Know ___ No Response ___

Next, the E removes the gray clay ball from the S's hand and places the gray ball on the table 8-10 inches in front of the hand in which it was held. Then the red clay ball is removed and placed in the hand opposite the one in which it originally appeared. Next the lighter gray clay ball is placed in the remaining empty hand, so the S will know that the red ball is the heavier of the two. The S also is required to verbalize this latter fact.

DOES ONE OF THE CLAY BALLS WEIGH MORE?

Yes ___ No ___ I Don't Know ___ No Response ___

(If "Yes," then) WHICH ONE?

Red ___ Gray ___ I Don't Know ___ No Response ___

Ordinality--Weight (Cont.)

The gray clay ball is removed and placed on the table 8-10 inches in front of the hand in which it was held. Finally, the E removes the red clay ball from the table, and asks the following questions:

(a) DO THESE TWO CLAY BALLS WEIGH THE SAME?

Yes ___ No ___ I Don't Know ___ No Response ___

(b) DOES ONE OF THE CLAY BALLS WEIGH MORE?

Yes ___ No ___ I Don't Know ___ No Response ___

(If "Yes," then) WHICH ONE?

Heavy ___ Light ___

(c) DOES ONE OF THE CLAY BALLS WEIGH LESS?

Yes ___ No ___ I Don't Know ___ No Response ___

(If "Yes," then) WHICH ONE?

Light ___ Heavy ___

Arithmetic Assessment

Materials:

16 sheets of paper for Ks with one of 16 incomplete addition equations printed on each. 16 additional sheets of paper for all others with subtraction equations.

Procedure:

For Ks - Ss are told that they are to add up some numbers.
For all others - Ss are told that they are to add up some numbers and subtract some other numbers.

All Ss are allowed 1 1/2 mins. to complete the addition equations, and 1 1/2 mins. to complete the subtraction equations.

A sheet is placed in front of S that will correspond with the questions below.

HOW MANY APPLES IS 1 APPLE PLUS 1 APPLE?

1 + 2
1 + 3
1 + 4
2 + 3
4 + 2
2 + 2
2 + 1
2 + 4
3 + 4
3 + 3
3 + 1
3 + 2
4 + 3
4 + 4
4 + 1

HOW MANY APPLES IS 2 APPLES MINUS 1 APPLE?

3 - 1
5 - 2
8 - 4
4 - 3
4 - 1
6 - 2
2 - 1
5 - 3
5 - 1
7 - 4
3 - 2
5 - 4
6 - 4
7 - 3
8 - 4

RAW SCORE DATA

Task performance, task level, and assigned concept stage for each subject by grade are presented in this section in table form. Each row represents a subject identified by subject number. Column headings reflect the following:

1--Seriation--ordering of sticks
2--Seriation--ordering of circles
3--Seriation--prediction
4--Seriation--placement
5--Serial Correspondence--stick-to-circle
6--Serial Correspondence--extension, compression, and scrambled cases
7--Dichotomous Sorting
8--Class Inclusion A--order of presentation
9--Class Inclusion A
10--Class Inclusion B
11--Conservation of Number--prediction
12--Conservation of Number--deformation
13--Unit--Cards A and D
14--Unit--Cards B and C
15--Unit--Cards E and F
16--Seriation Level
17--Correspondence Level
18--Relations Stage
19--Sorting Level
20--Class Inclusion Level
21--Class Stage
22--Conservation of Number Level
23--Unit Level
24--Number Stage
25--Arithmetic Proficiency Scores
26--Arithmetic Stage
27--Ordinality (number of series correct)
28--Ordinality Stage
29--Cardinality Stage

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	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
24	7	9	3	3	10	3	1	1	1	2	2	1	0	0	0	2	1	I	2	1	I	1	1	I	0	I	1	II	I
25	7	8	3	3	10	5	2	1	2	3	3	0	0	1	0	1	2	I	2	1	II	1	1	I	3	I	2	III	I
26	6	8	2	2	10	3	0	1	3	4	0	0	0	1	0	1	1	I	1	2	I	1	1	I	0	I	0	I	I
27	5	9	1	3	10	2	1	2	1	3	0	0	0	0	0	1	1	I	2	1	I	1	1	I	0	I	1	II	I
28	5	10	3	3	10	9	1	1	0	1	1	0	0	0	0	1	3	I	2	1	I	1	1	I	13	III	2	III	I
29	5	3	3	3	10	4	1	2	0	4	3	1	0	0	0	1	1	I	2	1	I	2	1	I	16	III	2	III	I
30	5	6	0	1	10	2	0	1	2	2	1	1	0	0	1	1	1	I	1	1	I	1	1	I	4	I	1	II	I
31	1	4	1	1	6	1	1	2	0	0	1	1	0	1	0	1	1	I	2	1	I	1	1	I	0	I	1	II	I
33	7	10	0	1	7	5	0	1	2	2	1	0	0	0	0	1	1	I	1	1	I	1	1	I	0	I	2	III	I
35	5	4	0	2	2	1	0	1	1	2	1	1	0	0	0	1	1	I	1	1	I	1	1	I	4	I	0	I	I
36	3	0	0	1	2	3	0	2	0	0	1	1	0	0	0	1	1	I	1	1	I	1	1	I	0	I	2	III	I
37	1	0	0	3	10	2	0	1	3	3	1	0	0	0	0	1	1	I	1	2	I	1	1	I	6	II	1	II	I
38	7	10	0	0	10	6	0	1	0	0	3	3	1	2	0	1	2	I	1	1	I	3	1	II	14	III	0	I	I
39	5	2	2	3	10	5	1	1	2	1	2	2	0	0	0	1	2	I	2	1	I	2	1	I	2	I	2	III	I
40	3	2	0	0	10	3	0	2	1	3	0	0	0	0	0	1	1	I	1	1	I	1	1	I	0	I	2	III	I
41	3	9	0	3	3	4	1	2	4	4	2	0	0	1	0	1	1	I	2	2	II	1	1	I	5	I	2	III	I
42	4	0	0	1	8	1	0	1	1	2	1	0	0	1	0	1	1	I	1	1	I	1	1	I	0	I	0	I	I
43	6	8	0	3	10	2	1	1	1	1	1	0	1	0	0	1	1	I	2	1	I	1	1	I	4	I	2	III	I
44	3	7	1	0	10	1	1	1	2	3	0	0	2	0	0	1	1	I	2	1	II	1	1	I	6	II	2	III	I
45	3	6	1	3	10	2	2	2	4	2	3	3	0	2	0	1	1	I	2	2	II	3	1	I	0	I	1	II	I

ERIC

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	[illegible]	22	23	24	25	26	27	28	29
45	7	10	0	3	10	9	3	2	4	4	3	3	2	2	2	2	3	II	3	2	II	3	3	III	32	III	1	II	III
46	7	10	3	3	10	7	3	1	3	4	3	3	2	2	0	3	2	II	3	2	II	3	2	II	32	III	2	III	II
47	7	10	2	3	10	9	2	2	0	1	3	3	0	1	0	2	3	II	2	1	I	3	1	I	28	III	0	I	I
48	7	10	0	0	10	8	3	1	4	5	2	3	1	1	1	1	2	I	3	3	III	2	2	II	32	III	2	III	II
49	7	10	3	3	10	6	1	1	4	2	3	3	2	2	0	3	2	II	2	2	II	3	2	II	31	III	2	III	II
50	7	10	3	3	10	8	1	1	2	3	3	3	2	1	1	3	2	II	2	1	II	3	2	II	31	III	2	III	II
51	7	10	3	3	10	7	2	1	4	4	3	3	2	2	2	3	2	II	2	2	II	3	3	III	32	III	2	III	III
53	7	10	3	3	10	9	3	1	4	5	3	3	2	2	2	3	3	III	3	3	III	3	3	III	31	III	2	III	III
54	7	10	3	3	10	8	2	2	4	5	3	3	1	2	2	3	2	II	2	3	II	3	2	II	27	II	1	II	II
56	7	10	3	3	10	6	1	2	3	5	3	3	2	2	2	3	2	II	2	3	II	3	2	II	32	III	2	III	II
57	7	10	3	3	10	9	2	2	4	3	3	3	1	2	2	3	3	III	2	2	II	3	2	II	32	III	2	III	II
58	7	10	3	3	10	8	2	1	4	5	3	3	2	2	2	3	2	II	2	3	II	3	3	III	32	III	2	III	III
59	7	10	3	3	10	5	2	2	2	2	3	3	2	1	1	3	2	II	2	1	II	3	2	II	27	II	1	II	II
60	7	10	3	3	10	5	1	2	0	3	0	2	1	1	0	3	2	II	2	1	I	1	1	I	32	III	2	III	I
61	7	10	2	3	10	8	3	1	4	5	3	3	2	2	2	2	2	II	3	3	III	3	3	III	32	III	1	II	III
62	7	10	0	3	10	8	1	2	3	3	3	0	2	0	0	2	2	II	2	2	II	1	1	I	16	I	2	III	I
63	7	10	3	3	10	9	1	2	0	2	3	3	2	1	2	3	3	III	2	1	I	3	2	II	31	III	1	II	II
67	7	10	3	3	10	6	2	1	4	3	3	3	2	2	2	3	2	II	2	2	II	3	3	III	26	II	2	III	III
96	7	10	3	3	10	9	3	2	4	5	3	3	2	2	2	3	3	III	3	3	III	3	3	III	32	III	2	III	III
101	7	10	3	3	10	9	2	2	2	4	3	3	2	2	2	3	3	III	2	2	II	3	3	III	32	III	2	III	III

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